A proof of Finite Family Developments for Higher-Order Rewriting using a Prefix Property

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Abstract. A prefix property is the property that, given a reduction, the ancestor of a prefix of the target is a prefix of the source. In this paper we prove a prefix property for the class of Higher-Order Rewriting Systems with patterns (HRSs), by reducing it to a similar prefix property of a λ-calculus with explicit substitutions. This prefix property is then used to prove that Higher-order Rewriting Systems enjoy Finite Family Developments. This property states, that reductions in which the creation depth of the redexes is bounded are finite, and is a useful tool to prove various properties of HRSs.

1 Introduction

Higher-order Rewriting Systems (HRSs), as introduced by Nipkow in 1991 [12,10], are a powerful tool to study the metatheory of declarative programming languages, like λProlog and Haskell, on the one hand, and theorem provers and proof checkers, like Isabelle, on the other. Also, many (extensions of) λ-calculi can be encoded as instances of HRSs, so that results obtained for HRSs carry over to other interesting domains.

In this paper, we prove two properties of HRSs where left-hand sides of rule are restricted to be patterns. First we prove a prefix property, by reducing this property to a similar prefix property for a λ-calculus with explicit substitutions. The prefix property says that, given a step, the ancestor of a prefix of the target is a prefix of the source. Consider, as an example, the (first-order) rewriting system with the single rule

$$\mathsf{f}(x) \to \mathsf{g}(\mathsf{f}(x), x)$$

and the step $\mathsf{f}(\mathsf{h}(\mathsf{a})) \to \mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{a})), \mathsf{h}(\mathsf{a}))$. Now, $p = \mathsf{g}(\mathsf{f}(\square), \mathsf{h}(\square))$ is a prefix of the target. Intuitively, its ancestor is $\mathsf{f}(\mathsf{h}(\square))$, because

$$s = \mathsf{f}(\mathsf{h}(\square)) \to \mathsf{g}(\mathsf{f}(\mathsf{h}(\square)), \mathsf{h}(\square)) = t$$

and p is contained in t. And indeed, $\mathsf{f}(\square)$ is a prefix of the source.

Many different prefix properties are possible: we can, e.g., vary in how the notions of prefix and ancestor are formalized, and we may impose additional

conditions on the form of the prefixes. Prefix properties are already known for first-order TRSs [2,14] and (a labelling of) the λ-calculus with β-reductions [2], and have many applications, such as (head) needed reductions [14, Chap. 8] and normalization of outermost-fair reductions [14, Chap. 9]. A similar property is proved in Van Daalen's Square Brackets Lemma [15].

The second contribution is that we prove Finiteness of Family Developments (FFD) for HRS, by reducing this property to the prefix property described above. FFD states that each reduction, in which the "creation depth", or *family*, of function symbols is bounded, is finite. The intuition behind the notion of family is that in a step $C[l^\sigma] \to C[r^\sigma]$, the symbols of r depend on the symbols of l, and therefore have a higher creation depth, while the symbols in C and σ do not take part in the step and have the same creation depth in both source and target. For example, consider the following infinite reduction, using the rewrite system above. We label the function symbols with their creation depth.

$$\mathsf{f}^0(\mathsf{a}^0) \to \mathsf{g}^1(\mathsf{f}^1(\mathsf{a}^0), \mathsf{a}^0) \to \mathsf{g}^1(\mathsf{g}^2(\mathsf{f}^2(\mathsf{a}^0), \mathsf{a}^0)) \to \mathsf{g}^1(\mathsf{g}^2(\mathsf{g}^3(\mathsf{f}^3(\mathsf{a}^0), \mathsf{a}^0), \mathsf{a}^0)) \to \cdots$$

Clearly, in this infinite reduction, the creation depth of the f's grows without bound. FFD states that restricting the creation depth to a finite number, yields finite reductions. FFD is a useful tool to prove various properties of rewrite systems, such as termination (e.g. termination of simply typed λ-calculus follows from FFD, see Appendix A), existence of standard reductions, etc.

Note. This is an extended and more detailed version of [4]. Because this technical report includes additional definitions and lemmas, the numbering of examples, definitions, theorems and lemmas may differ in some places.

2 Preliminaries

We presuppose knowledge of the simply typed λ-calculus. Here we give a short overview of Higher-Order Rewrite Systems (HRSs) [10]. In particular, we consider HRSs as HORSs [16] with the simply typed λ-calculus as substitution calculus. We refer to [14, Sect. 11.2] for a good introduction.

Simple types are generated from a set of base types by the type constructor $\to$. Let Σ be a signature of simply typed function symbols. We define a *preterm* to be a simply typed λ-terms over Σ. We want to consider $\beta\eta$-equivalence classes of preterms. Since it is well known that β-reduction, combined with restricted η-expansion ($\overline{\eta}$-reduction), is confluent and terminating, we take $\beta\overline{\eta}$-normal forms as unique representatives of the $\beta\overline{\eta}$-equivalence classes. We define: a *term* is a preterm in $\beta\overline{\eta}$-normal form. In the following, s, t will range over terms (and, whenever indicated, over preterms as well).

A sequence $a_1, \ldots, a_n$ will sometimes be written as $\overline{a_n}$, or just $\overline{a}$ if the length is not important or clear from the context. Juxtaposition of two sequences denotes concatenation.

For terms or preterms $s, t_1, \ldots, t_n$, we write $s(t_1, \ldots, t_n)$ for $st_1 \cdots t_n$, or, in the case of terms, the β-normal form thereof. We also introduce the shorthand $\lambda \overline{x_n}.s$ for $\lambda x_1. \ldots .\lambda x_n.s$. With $FV(s)$ we denote the set of free variables of term or preterm s, and with $Sym(s)$ the set of function symbols present in the term or preterm. If $\lambda \overline{x}.a(\overline{s})$ is a term, then a is called the *head* of that term (a is a function symbol or variable).

In the class of HRSs that we consider, the left-hand sides of rules are restricted to be *local patterns*. For patterns, unification is decidable and unique most general unifiers exist [11]. Local patterns, additionally, are linear (each free variable occurs at most once) and fully-extended (free variable have all bound variables in scope as argument). These extra requirements have a similar purpose as the requirement of left-linearity in first-order TRS: they keep matching local. To match a non-linear pattern, it is possible that subterms outside the pattern need to be checked for equality; to match a non fully extended pattern, it is possible that such a subterm must be checked for the non-occurrence of a variable. Because the notion of pattern depends on what the free variables are, we need to parametrize the notion with a context of variables, and obtain the following inductive definition:

Definition 2.1 (Pattern). *Let $\overline{x}$ be a sequence of variables.*

  (i) *A term s is an $\overline{x}$-pattern if:*
    − *$s = a(s_1, \ldots, s_n)$ and either $a \in \overline{x} \cup \Sigma$ and $s_1, \ldots, s_n$ are $\overline{x}$-patterns; or $s_1, \ldots, s_n$ is η-equivalent to a sequence of distinct variables from $\overline{x}$.*
    − *$s = \lambda y.s_0$ and s_0 is an $\overline{x}y$-pattern.*
  (ii) *A term s is* linear outside $\overline{x}$, *if each free variable which is not in $\overline{x}$, occurs in it at most once. A term s is a* fully extended $\overline{x}$-pattern, *if, in the second case of the above definition, $s_1, \ldots, s_n =_\eta \overline{x}$. A term s is a* local $\overline{x}$-pattern, *if s is linear outside $\overline{x}$ and a fully extended $\overline{x}$-pattern.*

Examples of local patterns are $f(x)$, $g(\lambda xy.f(z(x,y)))$ and $h(\lambda x.z(x))$. Examples of non-local patterns are $g(\lambda xy.f(y))$ (not fully-extended) and $h(\lambda x.z(x), \lambda x.z(x))$ (not linear). An example of a non-pattern is $g(z(a))$.

In the following, p, q will range over patterns, and the word pattern (without the sequence of variables) will refer to a $\emptyset$-pattern.

Definition 2.2 (HRS). *A* rewrite rule *(for a signature Σ) is a pair $\lambda \overline{x}.l_0 \to \lambda \overline{x}.r_0$ of closed Σ-terms of the same type, such that $l_0 = f(s_1, \ldots, s_n)$ and l_0 is a local pattern not η-equivalent to a variable. An HRS is a tuple $\mathcal{H} = \langle \Sigma, R \rangle$, where Σ is a signature and R a set of rewrite rules for Σ.*

The rewrite relation $\to_\mathcal{H}$ *is defined as follows: $s \to_\mathcal{H} t$ if there exist a context C such that $s =_\beta C[l]$ and $t =_\beta C[r]$, for some rule $l \to r \in R$.*

For arbitrary rewrite system $\mathcal{R}$, we denote with $\twoheadrightarrow_\mathcal{R}$ the reflexive, transitive closure of $\to_\mathcal{R}$.

Note that there is no substitution in the definition of the rewrite relation, such as in first-order term rewriting systems (but see also Remark 2.4). The leading abstractions of the rules take the role of the substitution, as can be seen in the next example:

Example 2.3. Let the HRS $\mathcal{M}ap$, implementing the higher-order function map, be defined by:

$$\lambda z.\mathsf{map}(\lambda x.z(x), \mathsf{nil}) \to \lambda z.\mathsf{nil}$$
$$\lambda zuv.\mathsf{map}(\lambda x.z(x), \mathsf{cons}(u, v)) \to \lambda zuv.\mathsf{cons}(z(\mathsf{e}(u)), \mathsf{map}(\lambda x.z(x), v))$$

Here, cons and nil are the list constructors, viz. list composition and the empty list, respectively. The reason for the symbol e is to make the HRS non-collapsing (see Def. 2.5). A reduction of two $\mathcal{M}ap$-steps is the following:

$$
\begin{aligned}
&\mathsf{map}(\lambda x.\mathsf{f}(x), \mathsf{cons}(\mathsf{a}, \mathsf{nil})) \\
=_\beta \quad &(\lambda zuv.\mathsf{map}(\lambda x.z(x), \mathsf{cons}(u, v)))(\lambda x'.\mathsf{f}(x'), \mathsf{a}, \mathsf{nil}) \\
\to_{\mathcal{M}ap} \quad &\underline{(\lambda zuv.\mathsf{cons}(z(\mathsf{e}(u)), \mathsf{map}(\lambda x.z(x), v)))}(\lambda x'.\mathsf{f}(x'), \mathsf{a}, \mathsf{nil}) \\
=_\beta \quad &\mathsf{cons}(\mathsf{f}(\mathsf{e}(\mathsf{a})), \mathsf{map}(\lambda x.\mathsf{f}(x), \mathsf{nil})) \\
=_\beta \quad &\mathsf{cons}(\mathsf{f}(\mathsf{e}(\mathsf{a})), (\lambda z.\mathsf{map}(\lambda x.z(x), \mathsf{nil}))(\lambda x'.\mathsf{f}(x'))) \\
\to_{\mathcal{M}ap} \quad &\mathsf{cons}(\mathsf{f}(\mathsf{e}(\mathsf{a})), \underline{(\lambda z.\mathsf{nil})}(\lambda x'.\mathsf{f}(x'))) \\
=_\beta \quad &\mathsf{cons}(\mathsf{f}(\mathsf{e}(\mathsf{a})), \mathsf{nil})
\end{aligned}
$$

Note how the (underlined) left-hand sides are *literally* replaced by the (also underlined) right-hand sides.

In later examples, the leading abstractions of rewrite rules will be omitted; in other words, we will write $l \to r$ for $\lambda\overline{x}.l \to \lambda\overline{x}.r$.

Substitutions are maps from variables to terms. Application of a substitution $\sigma = [x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$ to a term s is defined as: $s^\sigma = (\lambda x_1 \ldots x_n.s)t_1 \ldots t_n$ (where this term is, as always, implicitly reduced to $\beta\overline{\eta}$-normal form). In the following, $\rho, \sigma, \tau, \upsilon$ will range over substitutions. The composition of substitutions σ and τ is denoted by $\sigma\,;\tau$, where $s^{\sigma;\tau} = (s^\sigma)^\tau$. A substitution is called linear, if each free variable occurs in its codomain at most once, i.e. if all terms of its codomain are linear and have mutually disjoint free variables. A (fully extended) $\overline{x}$-pattern substitution is a substitution of which the codomain consists of (fully extended) $\overline{x}$-patterns.

Remark 2.4. The rewrite relation of Def. 2.2 can alternatively, and more in the fashion of first-order TRSs, be defined in the following way:

$$s \to_{\mathcal{H}} t \quad \text{if} \quad s =_\beta C[l_0^\sigma] \text{ and } t =_\beta C[r_0^\sigma]$$

where $\lambda\overline{x}.l_0 \to \lambda\overline{x}.r_0 \in R$ and σ is a substitution with $Dom(\sigma) = \overline{x}$. This alternative definition, however, requires the notion of substitution to be defined, and therefore we prefer the other one. In the rest of the paper, we will sometimes implicitly switch definitions.

4

Intuitively, a rewrite rule is collapsing, if it can bring context and subtitution, or different parts of the substitution, together, i.e. if, after the application of the rule, a function symbol of the context can be directly connected to a function symbol of the substitution. This can happen in two specific cases, which we will use as a definition:

Definition 2.5. *A term s is* collapsing*, if one of the following applies:*

- *(context-subst): $s = x(s_1, \ldots, s_n)$, where x is a free variable; or*
- *(subst-subst): $s = C[x(s_1, \ldots, s_n)]$, and for some k, $s_k = \lambda \overline{z}.y(t_1, \ldots, t_m)$, where C is a context, x is a free variable, and y a free or bound variable.*

A rewrite rule $\lambda \overline{x}.l \to \lambda \overline{x}.r$ is collapsing, if r is collapsing, and an HRS is collapsing, if at least one of its rules is.

Example 2.6.

- The rules $\lambda x.\mathsf{f}(x) \to \lambda x.x$ and $\lambda z.\mathsf{mu}(\lambda x.z(x)) \to \lambda z.z(\mathsf{mu}(\lambda x.z(x)))$ are collapsing due to the (context-subst) condition. Suppose that $\mathsf{g}(\mathsf{a})$ is a redex, and consider the step $\mathsf{g}(\mathsf{f}(\mathsf{a})) \to_{\mathcal{H}} \mathsf{g}(\mathsf{a})$. This step creates the redex $\mathsf{g}(\mathsf{a})$ without creating one of its function symbols.
- The rule $\lambda yz.\mathsf{g}(\lambda x.z(x), y) \to \lambda yz.\mathsf{f}(z(y))$ is collapsing due to the (subst-subst) condition. Supppose that $\mathsf{f}(\mathsf{a})$ is a redex. Then the step $\mathsf{g}(\lambda x.\mathsf{f}(x), \mathsf{a}) \to \mathsf{f}(\mathsf{a})$ creates the redex $\mathsf{f}(\mathsf{a})$ without creating one of its function symbols.
- The rule $\lambda yz.\mathsf{app}(\mathsf{lam}(\lambda x.z(x)), y) \to \lambda yz.z(y)$ is collapsing due to both the (context-subst) and the (subst-subst) conditions.

An HRS $\mathcal{H}$ is *terminating*, if there are no infinite rewrite chains, i.e. if the inverse of its rewrite relation $\twoheadrightarrow_{\mathcal{H}}$ is well-founded. Let $\subseteq$ be the subterm relation, i.e. the reflexive, transitive closure of the smallest relation $\subseteq_1$ such that $s_1 \subseteq_1 s_1 s_2$, $s_2 \subseteq_1 s_1 s_2$, $s \subseteq_1 \lambda x.s$. We define: $\ll_{\mathcal{H}} = {\leftarrow}_{\mathcal{H}} \cup \subseteq$.

Lemma 2.7. *$\ll_{\mathcal{H}}$ is well-founded if and only if $\mathcal{H}$ is terminating.*

Proof. Corresponds to [17, Lemma 7 (3)]. □

3 Labelling HRSs with natural numbers

Labelling rewriting systems is a well-known method to formalize the notion of redex family; see e.g. [8,9]. In this section, we develop a labelling, in the sense of [18,14], for HRSs, analogous to the labelling for the λ-calculus used by Hyland [6] and Wadsworth [19]. Each function symbol is labelled by a natural number, representing the "creation depth" of the function symbol, and the rules are labelled such that every function symbol of the right-hand side is labelled with the largest label of the left-hand side plus one.

Definition 3.1 (ω-labelling).

(i) *The ω-labelling of a signature Σ is defined as: $\Sigma^\omega = \{f^\ell \mid f \in \Sigma, \ell \in \mathbb{N}\}$.*

(ii) *The family of a term s, denoted* $\mathrm{fam}(s)$, *is the largest label of s, i.e.:*

$$\mathrm{fam}(s) = \max\{\ell \mid f^\ell \in Sym(s)\}$$

(iii) *Let s be a term, and $\ell \in \mathbb{N}$ a label. Then:*

$$x(s_1, \ldots, s_n)^\ell = x(s_1^\ell, \ldots, s_n^\ell)$$
$$f(s_1, \ldots, s_n)^\ell = f^\ell(s_1^\ell, \ldots, s_n^\ell)$$
$$(\lambda x.s_0)^\ell = \lambda x.s_0^\ell$$

(iv) *The projection operation $|\cdot|_\omega$ is the mapping from Σ^ω to Σ given by $|f^\ell|_\omega = f$. The mapping is homomorphically extended to terms.*

(v) *Let $\mathcal{H} = \langle \Sigma, R \rangle$. The ω-labelled version of $\mathcal{H}$ is defined as: $\mathcal{H}^\omega = \langle \Sigma^\omega, R^\omega \rangle$, where R^ω consist of all rules $l' \to r^{(\mathrm{fam}(l')+1)}$ such that $l \to r \in R$ and $|l'|_\omega = l$.*

The ω-labelling only labels *function symbols*, not variables, abstractions or applications. The reason for this is that we want the ω-labelling of an HRS to be an HRS itself (otherwise it would not be a labelling in the sense of [18,14]). Labelling variables is impossible, because α-equivalent terms are identified. Labelling abstractions and applications is impossible because we have fixed the (unlabelled) simply typed λ-calculus as substitution calculus.

Example 3.2. The labelled HRS $\mathcal{M}ap^\omega$ consists, among others, of the rules:

$$\mathsf{map}^0(\lambda x.z(x), \mathsf{nil}^0) \to \mathsf{nil}^1$$
$$\mathsf{map}^1(\lambda x.z(x), \mathsf{nil}^1) \to \mathsf{nil}^2$$
$$\mathsf{map}^0(\lambda x.z(x), \mathsf{cons}^0(u, v)) \to \mathsf{cons}^1(z(\mathsf{e}^1(u)), \mathsf{map}^1(\lambda x.z(x), v))$$
$$\mathsf{map}^0(\lambda x.z(x), \mathsf{cons}^1(u, v)) \to \mathsf{cons}^2(z(\mathsf{e}^2(u)), \mathsf{map}^2(\lambda x.z(x), v))$$

A labelled reduction corresponding to the reduction of Ex. 2.3 is the following:

$$\mathsf{map}^0(\lambda x.\mathsf{f}^0(x), \mathsf{cons}^0(\mathsf{a}^0, \mathsf{nil}^0))$$
$$=_\beta \quad (\lambda zuv.\mathsf{map}^0(\lambda x.z(x), \mathsf{cons}^0(u, v)))(\lambda x'.\mathsf{f}^0(x'), \mathsf{a}^0, \mathsf{nil}^0)$$
$$\to_{\mathcal{M}ap} \quad \underline{(\lambda zuv.\mathsf{cons}^1(z(\mathsf{e}^1(u)), \mathsf{map}^1(\lambda x.z(x), v)))}(\lambda x'.\mathsf{f}^0(x'), \mathsf{a}^0, \mathsf{nil}^0)$$
$$=_\beta \quad \mathsf{cons}^1(\mathsf{f}^0(\mathsf{e}^1(\mathsf{a}^0)), \mathsf{map}^1(\lambda x.\mathsf{f}^0(x), \mathsf{nil}^0))$$
$$=_\beta \quad \mathsf{cons}^1(\mathsf{f}^0(\mathsf{e}^1(\mathsf{a}^0)), (\lambda z.\mathsf{map}^1(\lambda x.z(x), \mathsf{nil}^0))(\lambda x'.\mathsf{f}^0(x')))$$
$$\to_{\mathcal{M}ap} \quad \mathsf{cons}^1(\mathsf{f}^0(\mathsf{e}^1(\mathsf{a}^0)), \underline{(\lambda z.\mathsf{nil}^2)}(\lambda x'.\mathsf{f}^0(x')))$$
$$=_\beta \quad \mathsf{cons}^1(\mathsf{f}^0(\mathsf{e}^1(\mathsf{a}^0)), \mathsf{nil}^2)$$

Notice how only the labels of function symbols involved in the step (i.e. the underlined ones) are increased.

The following two lemmas provide a correspondence between labelled and unlabelled reductions:

Lemma 3.3. *Let $\mathcal{H}$ be an HRS. $\mathcal{H}^\omega$ is orthogonal/collapsing/erasing, if and only if $\mathcal{H}$ is.*

Proof. By induction on the length we easily prove that s^ℓ is collapsing, if and only if s is. $\qquad\square$

Lemma 3.4. *Let $\mathcal{H}$ be an HRS.*

(i) *If $s \rightarrow_\mathcal{H} t$, then, for each s' such that $|s'|_\omega = s$, there is a t' such that $s' \rightarrow_{\mathcal{H}^\omega} t'$ and $|t'|_\omega = t$.*

(ii) *If $s \rightarrow_{\mathcal{H}^\omega} t$, then $|s|_\omega \rightarrow_\mathcal{H} |t|_\omega$.*

Proof. Both items are easily proved by induction on the context of the step. $\quad\square$

4 The Prefix Property

We call p a prefix of term t, if it is a pattern, and there exists a substitution σ such that $p^\sigma = t$. Given a step $s \rightarrow t$, a subterm q of s is the ancestor of a subterm p of t, if the symbols of t "trace to" the symbols of s. This notion is formalized here using labelling together with the rewrite relation: q is an ancestor of p, if $\mathrm{fam}(p) \geq \mathrm{fam}(q)$ and $q \twoheadrightarrow_{\mathcal{H}^\omega} p^\upsilon$. The substitution υ is necessary because q might reduce to a "bigger" term than p; typically, υ has only function symbols which are also in p. Using these formalizations, we prove in this section the following theorem (proof begins on page 15).

Theorem 4.1 (Prefix Property). *Let $\mathcal{H}^\omega$ be the ω-labelling of a non-collapsing HRS, s a term, p a local $\overline{x}$-pattern and σ a substitution. If $s \rightarrow_{\mathcal{H}^\omega} p^\sigma$, then there exist a local $\overline{x}$-pattern q and a substitution τ, such that $s = q^\tau$, $\mathrm{fam}(p) \geq \mathrm{fam}(q)$, and either:*

$-$ *$q \rightarrow_{\mathcal{H}^\omega} p^\upsilon$, for some substitution υ such that $\upsilon\,;\tau = \sigma$; or* $\qquad$ *(trm)*

$-$ *$q = p$ and $\tau \rightarrow_{\mathcal{H}^\omega} \sigma$.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ *(sub)*

The theorem states that, given a prefix of the target, its ancestor is a prefix of the source. There are two possibilities: either the prefix takes part in the step, or the step occurred fully in the substitution. Note that, in the first case, we do not only require that its ancestor is a prefix, but also that the suffix stays the same (except for duplicated subterms). In this regard, the lemma is stronger than e.g. the prefix property (for the λ-calculus) proved in [2, Prop. 7.4].

Example 4.2. Consider the following $\mathcal{M}ap^\omega$-step (see page 4):

$$\mathsf{h}^1(\mathsf{map}^3(\lambda x.\mathsf{f}^2(x), \mathsf{cons}^2(\mathsf{a}^5, \mathsf{nil}^1)) \rightarrow \mathsf{h}^1(\mathsf{cons}^4(\mathsf{f}^2(\mathsf{e}^4(\mathsf{a}^5)), \mathsf{map}^4(\lambda x.\mathsf{f}^2(x), \mathsf{nil}^1)))$$

First, let the prefix $p = \mathsf{h}^1(\mathsf{cons}^4(\mathsf{f}^2(y_1), y_2))$ of the target be given. The suffix is then given by $\sigma = [y_1 \mapsto \mathsf{e}^4(\mathsf{a}^5), y_2 \mapsto \mathsf{map}^4(\lambda x.\mathsf{f}^2(x), \mathsf{nil}^1)]$. Then:

$$q = \mathsf{h}^1(\mathsf{map}^3(\lambda x.\mathsf{f}^2(x), \mathsf{cons}^2(y_1, z_2)))$$
$$\upsilon = [y_1 \mapsto \mathsf{e}^4(z_1), y_2 \mapsto \mathsf{map}^4(\lambda x.\mathsf{f}(x), z_2)]$$
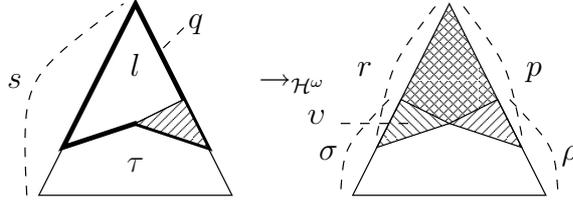$$\tau = [z_1 \mapsto \mathsf{a}^5, z_2 \mapsto \mathsf{nil}^1]$$

Fig. 1. The interesting case in the proof of the Prefix Property for HRSs

satisfy the conditions of the (trm) case. Second, let $p = \mathsf{h}(y)$ and $\sigma = [y \mapsto \mathsf{cons}^4(\mathsf{f}^2(\mathsf{e}^4(\mathsf{a}^5)), \mathsf{map}^4(\lambda x.\mathsf{f}^2(x), \mathsf{nil}^1)))]$. Then:

$$q = \mathsf{h}^1(y) \quad \text{and} \quad \tau = [y \mapsto \mathsf{map}^3(\lambda x.\mathsf{f}^2(x), \mathsf{cons}^2(\mathsf{a}^5, \mathsf{nil}^1))]$$

satisfy the conditions of the (sub) case.

The interesting case in the proof of the Prefix Property is the case that the step $s \to_{\mathcal{H}^\omega} p^\sigma$ occurs at the head. In this case we have that $s = l^\rho$ and $p^\sigma = r^\rho$, for some rule $l \to r$ and substitution ρ. This situation is depicted in Fig. 1. We want to construct an ancestor q that satisfies the (trm) case. It makes sense to try to do this by adding to the pattern l the parts of p that are not in r. However, due to the implicit β-conversions, these "parts of p that are not in r" are not easily obtained. The key observation is that the β-reduction from p^σ to normal form is a variable renaming, because p is a pattern and has only bound variables as arguments of free variables. The trick is to translate the prefix and suffix in such a way, that the variable renamings are already carried out (we need variable capturing, first-order substitutions for this, called *graftings*), trace the prefix back over the β-reduction from r^ρ to normal form, and find the prefix's ancestor, which is a prefix of r^ρ. Now, we are dealing with terms that are exactly equal, instead of only equal up to β-equality, and now the problem can be solved by using first-order unification techniques.

The above proof technique suggests that we need to prove a prefix property for β-reductions in the λ-calculus. This is difficult, however, since the λ-calculus does not cope well with graftings, because of the global nature of substitution. For example, let $C = (\lambda x.\square)\mathsf{a}$, $D = \square$ and $s = x$. Then $C \to_\beta D$, and $C[s] \to_\beta \mathsf{a}$, because the x in s is captured by the abstraction in the context and substituted for. However, $D[s] = x$ and thus $C[s] \not\to_\beta D[s]$. To tackle this problem, we use a λ-calculus with explicit substitutions, a variant of the λx-calculus, and prove a prefix property for it. Then, we simulate β-equality with this new calculus. In [5] a similar approach is taken w.r.t. higher-order unification.

4.1 The Prefix Property of the λx-Calculus

We use a variant of the λx-calculus [3], with explicit renamings. The calculus has both object variables (x, y, z) and metavariables (X, Y, Z). In the following,

we will refer to it simply by λx-calculus. The terms of the λx-calculus over some signature Σ are first-order terms given by the following grammar:

$$\Lambda_{\mathrm{x}} := x \mid X \mid f \mid \lambda x.\Lambda_{\mathrm{x}} \mid \Lambda_{\mathrm{x}}\Lambda_{\mathrm{x}} \mid \Lambda_{\mathrm{x}}\{x\backslash\Lambda_{\mathrm{x}}\}$$

where $f \in \Sigma$ and the object variables are considered as constants or names. M, N will range over λx-terms. Terms of the form $M\{x\backslash N\}$ will be called *explicit substitutions*, and the $\{x\backslash N\}$ part of an explicit substitution is called a *closure*. With $\mathrm{MV}(M)$ we will denote the set of metavariables of M, and with $Sym(M)$ the set of function symbols of M. The reduction rules of the λx-calculus are:

$$(\lambda x.M)N \to_{\mathrm{B}} M\{x\backslash N\}$$

$$
\begin{array}{ll}
x\{x\backslash N\} \to_{\mathrm{x}} N & (\lambda x.M)\{x\backslash N\} \to_{\mathrm{x}} \lambda x.M \\
y\{x\backslash N\} \to_{\mathrm{x}} y & (\lambda y.M)\{x\backslash N\} \to_{\mathrm{x}} \lambda z.M\{y\backslash z\}\{x\backslash N\} \\
f\{x\backslash N\} \to_{\mathrm{x}} f & (M_1 M_2)\{x\backslash N\} \to_{\mathrm{x}} M_1\{x\backslash N\}M_2\{x\backslash N\}
\end{array}
$$

where $x \neq y$ and z is a fresh object variable. The subcalculus x consists of all rules except the B-rule. The reduction relations $\to_{\mathrm{Bx}}$ and $\to_{\mathrm{x}}$ are the contextual closures of the above steps. Note that there is no reduction rule for terms of the form $X\{x\backslash N\}$, where X is a metavariable, and thus x-normal forms are characterized by the fact that sequences of closures are only applied to metavariables.

A λx-term is called *passive* if no metavariable X occurs in a subterm of the form $X\overline{\mu}(M_1, \ldots, M_n)$, where $\overline{\mu}$ is a sequence of closures; it is called *linear*, if every metavariable occurs in it at most once. In the following P, Q will range over linear, passive λx-terms.

Remark 4.3. It is well-known that the λx-calculus is *not* confluent on terms containing metavariables, as is witnessed by the following counterexample:

$$
\begin{array}{ll}
(\lambda x.(\lambda y.Z)Y)X & (\lambda x.(\lambda y.Z)Y)X \\
\twoheadrightarrow_{\mathrm{Bx}} (\lambda y'.Z\{y\backslash y'\}\{x\backslash X\})Y\{x\backslash X\} & \twoheadrightarrow_{\mathrm{Bx}} (\lambda x.Z\{y\backslash Y\})X \\
\twoheadrightarrow_{\mathrm{Bx}} Z\{y\backslash y'\}\{x\backslash X\}\{y'\backslash Y\{x\backslash X\}\} & \twoheadrightarrow_{\mathrm{Bx}} Z\{y\backslash Y\}\{x\backslash X\}
\end{array}
$$

At first sight, non-confluence seems problematic, because we're trying to use the λx-calculus to simulate the (confluent) λ-calculus. However, the translation to λ-calculus (see page 13) will remove all closures, and will project normal forms of the same λx-term to the same λ-term (modulo α-equivalence).

A *grafting* is a mapping from metavariables to λx-terms. The greek lowercase letters $\zeta, \eta, \theta, \kappa$ will range over graftings. Applying a grafting ζ to a term M, written $M[\zeta]$, is defined exactly as first order substitution, i.e.:

$$
\begin{array}{ll}
x[\zeta] = x & (\lambda x.M)[\zeta] = \lambda x.M[\zeta] \\
X[\zeta] = \zeta'(X) & (M_1 M_2)[\zeta] = M_1[\zeta]M_2[\zeta] \\
f[\zeta] = f & (M\{x\backslash N\})[\zeta] = M[\zeta]\{x\backslash N[\zeta]\}
\end{array}
$$

where $\zeta'(X) = \zeta(X)$, if $X \in Dom(\zeta)$, and $\zeta'(X) = X$, otherwise. A grafting is called *linear*, if every metavariable occurs in its codomain only once, i.e. its

codomain consists of linear λx-terms with mutually disjoint metavariables. A grafting is called *passive*, if all the terms of its codomain are passive.

Because λx-terms are first-order terms, unification is decidable. In the proof of the Prefix Property, we need the following property: if two λx-terms are unifiable, there exists a most general unifier (mgu). In fact, if we assume the unifiable terms to be linear and passive, then the mgu applied to one of the terms is a linear, passive λx-term again:

Lemma 4.4. *Let M, N be linear λx-terms, where $\mathrm{MV}(M) \cap \mathrm{MV}(N) = \emptyset$, and let ζ, η be graftings such that $M[\zeta] = N[\eta]$. There exist graftings ζ_0, η_0, κ such that $M[\zeta_0] = N[\eta_0]$, $\zeta_0 \, ; \kappa = \zeta$, $\eta_0 \, ; \kappa = \eta$, $Sym(\zeta_0) \subseteq Sym(N)$, $Sym(\eta_0) \subseteq Sym(M)$. Moreover, if M (N) is passive, then η_0 (ζ_0) consists of passive λx-terms.*

Proof. Since we consider λx-terms here as first-order terms, the (first part) of the lemma is essentially an instance of first-order unification.

By assumption, M and N are unified by $\langle \mu, \nu \rangle$ (the assumption that the metavariables are disjoint allows us to consider ζ and η together as a unifier), and thus have a most general unifier (mgu). We take $\langle \zeta_0, \eta_0 \rangle$ to be this mgu. The desired κ exists because any unifier is an instance of an mgu.

Assume $f \in Sym(\eta_0)$, but $f \notin Sym(M)$. Then it must be the case that $f \in Sym(\zeta_0)$. Since all the subterms of f must be in the graftings too, there exist graftings $\zeta_0', \eta_0', \kappa'$ such that $\zeta_0' \, ; \kappa = \zeta_0$ and $\eta_0' \, ; \kappa = \eta_0$. It follows by linearity that $\langle \zeta_0', \eta_0' \rangle$ is a unifier, contradicting the assumption. For the assumption that $f \in Sym(\zeta_0)$, but $f \notin Sym(N)$, a contradiction is derived through a symmetrical argument.

For the second part of the lemma, suppose M is passive, but $\eta_0(X)$ is not. Then $N[\eta_0]$ is not passive, and contains a subterm of the form $N_0 = X\overline{\mu}(N_1, \ldots, N_n)$. Because $M[\zeta_0] = N[\eta_0]$, so does $M[\zeta_0]$. There are two possibilities: if the root application of N_0 is in M, then M is not passive, contradicting the assumption; otherwise, N_0 must be a subterm of $\zeta_0(Y)$, for some $Y \in Dom(\mu_0)$, but then $\langle \zeta_0, \eta_0 \rangle$ is not a mgu. Again, a symmetrical arguments yields the desired result if N is passive, but $\mu_0(X)$ is not.

Example 4.5. Let:

$$M = \lambda x.\mathsf{g}(\mathsf{f}_1(X_1), X_2) \qquad N = \lambda x.\mathsf{g}(Y_1, \mathsf{f}_2(Y_2))$$
$$\zeta = [X_1 \mapsto \mathsf{a}, X_2 \mapsto \mathsf{f}_2(\mathsf{a})] \qquad \eta = [Y_1 \mapsto \mathsf{f}_1(\mathsf{a}), Y_2 \mapsto \mathsf{a}]$$

Then $M[\zeta] = \lambda x.\mathsf{g}(\mathsf{f}_1(\mathsf{a}), \mathsf{f}_2(\mathsf{a})) = N[\eta]$. We take $\zeta_0 = [X_2 \mapsto \mathsf{f}_2(Z_1)]$, $\eta_0 = [Y_1 \mapsto \mathsf{f}_1(Z_2)]$ and $\kappa = [Z_i \mapsto \mathsf{a}]$ to satisfy the conditions of the lemma.

In the next theorem, we prove the Prefix Property for the λx-calculus. P is a prefix of a λx-term M, if it is a linear, passive λx-term, and there exists a grafing ζ such that $P[\zeta] = M$. The notion of ancestor is again formalized using labelling and the rewrite relation; however, because we do not count creation depth in Bx-reductions, now the labels, or more generally, the function symbols of the prefix must be the same as those of its ancestor. Just like in Theorem 4.1, a prefix can either take part in the step, or not, resulting in two cases. Item (ii) is the extension of the Prefix Property to Bx-*reductions*.

Theorem 4.6 (λx-Prefix Property). *Let M be a closed λx-term, P a linear, passive λx-term and ζ a grafting.*

(i) *If $M \to_{\mathrm{Bx}} P[\zeta]$, then there exist a linear, passive λx-term Q and a grafting η such that $M = Q[\eta]$, $Sym(Q) = Sym(P)$ and either:*
   - $Q \to_{\mathrm{Bx}} P[\kappa]$ *where κ is some grafting such that $\kappa \,;\, \eta = \zeta$; or* (trm)
   - $Q = P$ *and $\eta \to_{\mathrm{x}} \zeta$.* (sub)

(ii) *If $M \twoheadrightarrow_{\mathrm{Bx}} P[\zeta]$, then there exist a linear, passive λx-term Q and a grafting η such that: $M = Q[\eta]$, $Sym(Q) = Sym(P)$, $Q \twoheadrightarrow_{\mathrm{Bx}} P[\kappa]$ where κ is some grafting such that $\kappa \,;\, \eta \twoheadrightarrow_{\mathrm{Bx}} \zeta$.*

Proof. (i) By induction on the context of the step $M \to_{\mathrm{Bx}} P[\zeta]$. In this proof, let id_ζ be the identity grafting on the domain of a grafting ζ, i.e. $\mathsf{id}_\zeta := [X \mapsto X \mid X \in Dom(\zeta)]$.

If $P = X$, then we take $Q = X$ and $\eta = \zeta[X \mapsto M]$, satisfying the (sub) case of the lemma. If the step does not occur at the head of the term, then the lemma follows simply from the induction hypothesis. Otherwise we look at the reduction rule which was applied (at the head).

The interesting case is when a closure is distributed over an application: i.e. $M = (M_1 M_2)\{x \backslash N\} \to_{\mathrm{x}} (M_1\{x \backslash N\})(M_2\{x \backslash N\}) = P[\zeta]$. Since P is not a metavariable, $P = P_1 P_2$, where $P_1[\zeta] = M_1\{x \backslash N\}$ and $P_2[\zeta] = M_2\{x \backslash N\}$. For P_2, there are two possible cases:

(a) If $P_2 = X$, for some metavariable X, then let $\kappa' = \mathsf{id}_\zeta \,;\, [X \mapsto Y\{x \backslash Z\}]$ and $\zeta' = \zeta[X \mapsto X, Y \mapsto M_2, Z \mapsto N]$, where Y, Z are fresh metavariables.
(b) Otherwise, let $\kappa' = \mathsf{id}_\zeta$ and $\zeta' = \zeta$.

In both cases, $P_2[\kappa'] = Q_2\{x \backslash R_2\}$, for some passive λx-terms Q_2 and R_2, and $\kappa' \,;\, \zeta' = \zeta$.

Since P is passive, we know that P_1 is not of the form $X\overline{\mu}$, where $\overline{\sigma}$ is a list of substitutions. In particular, $\overline{\mu}$ is not empty. Thus, $P_1 = Q_1\{x \backslash R_1\}$. Because P is linear, and the $\kappa'(X) = X$ for $X \in \mathrm{MV}(P_1)$, $P_1[\kappa'] = P_1$. Also, because in case (a), Y and Z are fresh metavariables, and in case (b), $\zeta' = \zeta$, it holds that $P_1[\zeta'] = P_1[\zeta]$.

Because P is linear, and R_1 and R_2 are different subterms of P, R_1 and R_2 are linear and have no metavariables in common. Since $R_1[\zeta'] = N = R_2[\zeta']$, we can apply Lemma 4.4, and obtain graftings ζ_1, ζ_2, η such that $R_1[\zeta_1] = R_2[\zeta_2]$, $\zeta_1 \,;\, \eta = \zeta'$, $\zeta_2 \,;\, \eta = \zeta'$, $Sym(\zeta_1) \subseteq Sym(R_2)$ and $Sym(\zeta_2) \subseteq Sym(R_1)$.

Let $R = R_1[\zeta_1]$ ($= R_2[\zeta_2]$). Because $Sym(\zeta_1) \subseteq Sym(R_2)$ and $Sym(\zeta_2) \subseteq Sym(R_1)$, $Sym(R) = Sym(R_1) \cup Sym(R_2)$. Next, let $\kappa_i = (\kappa' \,;\, \zeta_i){\restriction}\mathrm{FV}(P_i)$, for $i \in \{1, 2\}$. Since P is linear, κ_1 and κ_2 have disjoint domains. Let $\kappa = \kappa_1 \cup \kappa_2$. We define $Q = (Q_1 Q_2)\{x \backslash R\}$. Now:

$$Q = (Q_1 Q_2)\{x \backslash R\} \to_{\mathrm{x}} (Q_1\{x \backslash R\})(Q_2\{x \backslash R\}) = P_1[\kappa' \,;\, \zeta_1] P_2[\kappa' \,;\, \zeta_2] = P[\kappa]$$

Also, suppose $X \in Dom(\kappa)$. There are two subcases. If $X \in \mathrm{MV}(P_2)$, then $(\kappa \,;\, \eta)(X) = (\kappa' \,;\, \zeta_2 \,;\, \eta)(X) = (\kappa' \,;\, \zeta')(X) = \zeta(X)$. Otherwise, $\kappa'(X) = X$ and $\zeta'(X) = \zeta(X)$. So $\kappa \,;\, \eta = \zeta$.

Finally, $Sym(P) = Sym(Q_1) \cup Sym(Q_2) \cup Sym(R_1) \cup Sym(R_2) = Sym(Q_1) \cup Sym(Q_2) \cup Sym(R) = Sym(Q)$, yielding the result of the (trm) case.

(ii) By induction on the length of the reduction sequence $M \twoheadrightarrow_{\text{Bx}} P[\zeta]$. In the base case, if $M = P[\zeta]$, we take $Q = P$ and $\eta = \zeta$, satisfying the lemma with $\kappa = \emptyset$. Otherwise, let $M \twoheadrightarrow_{\text{Bx}} N \rightarrow_{\text{Bx}} P[\zeta]$. We apply item (i), obtaining a passive λx-term P' and grafting ζ' such that $M = P'[\zeta']$, $Sym(P') = Sym(P)$ and either the (trm) or (sub) case applies. Then we apply the induction hypothesis on the reduction $M \twoheadrightarrow_{\text{Bx}} P'[\zeta']$, obtaining a passive λx-term Q and grafting η such that $M = Q[\eta]$, $Sym(Q) = Sym(P')$ ($= Sym(P)$), $Q \twoheadrightarrow_{\text{Bx}} P'[\kappa]$ for some grafting κ such that $\kappa \, ; \eta \twoheadrightarrow_{\text{Bx}} \zeta'$. Now we distinguish the following cases:

- *(trm)* If $P' \rightarrow_{\text{Bx}} P[\kappa_0]$, for some κ_0 such that $\kappa_0 \, ; \zeta' = \zeta$, then $Q \twoheadrightarrow_{\text{Bx}} P'[\kappa] \twoheadrightarrow_{\text{Bx}} P[\kappa_0 \, ; \kappa]$ and $\kappa_0 \, ; \kappa \, ; \eta \twoheadrightarrow_{\text{Bx}} \kappa_0 \, ; \zeta' = \zeta$, as required.
- *(sub)* If $P' = P$ and $\zeta' \rightarrow_{\text{x}} \zeta$, then $Q \twoheadrightarrow_{\text{Bx}} P'[\kappa] = P[\kappa]$ and $\kappa \, ; \eta \twoheadrightarrow_{\text{Bx}} \zeta' \rightarrow_{\text{x}} \zeta$, as required. □

Example 4.7. Consider the Bx-reduction:

$$(\lambda x.\mathsf{g}(x, x))(\mathsf{f}(\mathsf{a})) \twoheadrightarrow_{\text{Bx}} \mathsf{g}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{a}))$$

and the prefix $P = \mathsf{g}(\mathsf{f}(X), Y)$ of the target. The suffix is $\zeta = [X \mapsto \mathsf{a}, Y \mapsto \mathsf{f}(\mathsf{a})]$. We can take:

- $Q = (\lambda x.\mathsf{g}(x, x))(\mathsf{f}(Y))$,
- $\kappa = [Y \mapsto \mathsf{f}(X)]$ and
- $\eta = [X \mapsto \mathsf{a}]$,

satisfying the conditions of Theorem 4.6 (ii).

4.2 Translating between Terms, Preterms and λx-Terms

We are now dealing with three types of terms: terms, preterms and λx-terms. In this subsection we develop the following translating operations between them:

- $\cdot^{\flat}$ and $\cdot^{\sharp}$ from terms to preterms and back;
- $\cdot^{\ominus}$ and $\cdot^{\oplus}$ from preterms to λx-terms and back.

Translating between terms and preterms. We introduce a pair of operations which interpret terms as preterms, and vice versa: with $s^{\flat}$ (s-flat), we denote (the unique representative of) the term s, interpreted as a preterm, and with $s^{\sharp}$ (M-sharp), we denote (the $\beta\overline{\eta}$-normal form of) the preterm s, interpreted as a higher-order term. These operations naturally generalize to substitutions. We will call a preterm s a (fully extended//linear/local) $\overline{x}$-prepattern, if it is in $\beta\overline{\eta}$-normal form, and $s^{\sharp}$ is a (linear/fully extended/local) $\overline{x}$-pattern.

Translating between preterms and λx-terms. We introduce the operations $\cdot\frac{\ominus}{\overline{x}}$ and $\cdot^{\oplus}$, which map λ-terms to λx-terms[1], and vice versa, as follows:

$$M^{\oplus} = (M{\downarrow_{\mathrm{x}}})_{\mathrm{N}}^{\oplus}$$

$$
\begin{aligned}
y_{\overline{x}}^{\ominus} &= Y \text{ if } y \notin \overline{x} & (Y\overline{\sigma})_{\mathrm{N}}^{\oplus} &= y \\
x_{\overline{x}}^{\ominus} &= x \ \text{ if } x \in \overline{x} & x_{\mathrm{N}}^{\oplus} &= x \\
f_{\overline{x}}^{\ominus} &= f & f_{\mathrm{N}}^{\oplus} &= f \\
(\lambda y.s)_{\overline{x}}^{\ominus} &= \lambda y.s_{\overline{x}y}^{\ominus} & (\lambda y.M)_{\mathrm{N}}^{\oplus} &= \lambda y.M^{\oplus} \\
(s_1 s_2)_{\overline{x}}^{\ominus} &= (s_1)_{\overline{x}}^{\ominus}(s_2)_{\overline{x}}^{\ominus} & (M_1 M_2)_{\mathrm{N}}^{\oplus} &= M_1^{\oplus} M_2^{\oplus}
\end{aligned}
$$

Note that $\cdot^{\oplus}$ also normalizes the term to x-normal form and removes explicit substitutions, and that, for each preterm s and sequence of variables $\overline{x}$, $(s_{\overline{x}}^{\ominus})^{\oplus} = s$. The operations above are naturally generalized to translations between substitutions and graftings.

Lemma 4.8. *Let M, N be λx-terms. $M \twoheadrightarrow_{\mathrm{Bx}} N$ if and only if $M^{\oplus} \twoheadrightarrow_{\beta} N^{\oplus}$.*

Proof. $(\Rightarrow)$ and $(\Leftarrow)$ are proved by induction on the length of the reductions $M \twoheadrightarrow_{\mathrm{Bx}} N$ and $M^{\oplus} \twoheadrightarrow_{\beta} N^{\oplus}$, respectively. □

Although the above lemma suggests that Bx-reduction in the λx-calculus can easily simulate β-reduction, there is still a problem: $\cdot^{\oplus}$ does not distribute properly over grafting application. The problem is similar to the problem given on page 8. Consider the λx-term $M := (\lambda x.\mathsf{f}(Y))\mathsf{a}$ and grafting $\zeta := [Y \mapsto x]$. Now $M[\zeta]^{\oplus} = (\lambda x.\mathsf{f}(x))\mathsf{a}$, $M^{\oplus} = (\lambda x.\mathsf{f}(y))\mathsf{a}$. $\zeta^{\oplus} = [y \mapsto x]$. Note that $(M^{\oplus})^{(\zeta^{\oplus})} = \lambda z.\mathsf{f}(x)$, because substitutions are capture-avoiding, and thus $M[\zeta]^{\oplus} \neq_{\beta} (M^{\oplus})^{(\zeta^{\oplus})}$.

The solution is to add as arguments to the free variables of the preterms as many (bound) variables as necessary (or more) to make the distribution work. In the example above we would have $s = (\lambda x.\mathsf{f}(y(x)))\mathsf{a}$ and $\sigma = [y \mapsto \lambda x.x]$. Now, s and σ are, in a way that will be formalized in the next definition, similar to M and ζ, but now $M[\zeta]^{\oplus} =_{\beta} s^{\sigma}$.

Definition 4.9. *Let M be a λx-term and ζ a grafting. A tuple $\langle s, \sigma \rangle$ of preterm and substitution is a λ-extension of $\langle M, \zeta \rangle$ if there are graftings θ_1, θ_2 such that:*

- *$s = M[\theta_1]^{\oplus}$ and $\sigma = (\theta_2 \, ; \zeta)^{\oplus}$;*
- *for each $X \in \mathrm{MV}(M)$, $\theta_1(X) = X(\overline{z})$ and $\theta_2(X) = \lambda\overline{z}.X$, where $\overline{z}$ is a list of variables containing at least the bound variables of M in scope that occur in $\zeta(X)$ (in arbitrary order).*

The notion of λ-extension is, again, naturally generalized to graftings and substitutions as the first component of the tuples.

Lemma 4.10. *Let $\langle s, \sigma \rangle$ be a λ-extension of $\langle M, \zeta \rangle$. Then:*

(i) $s^\sigma =_\beta M[\zeta]^\oplus$;
(ii) for each λx-term N such that $M \twoheadrightarrow_{\mathrm{Bx}} N$, $s^\sigma =_\beta N[\zeta]^\oplus$.

Proof. (i) Follows from the fact that for all $X \in \mathrm{MV}(M)$ it holds that

$$(\theta_1 \,; \theta_2)\!\downarrow_{\mathrm{x}}(X) = X\{z_1 \backslash z_1\} \cdots \{z_n \backslash z_n\}$$

and thus $s^\sigma =_\beta M[\theta_1 \,; \theta_2 \,; \zeta]^\oplus =_\beta M[\zeta]^\oplus$.

(ii) Assume that $M \twoheadrightarrow_{\mathrm{Bx}} N$. We take $t = N[\theta_1]^\oplus$ and show: (a) $s \twoheadrightarrow_\beta t$; and (b) $t^\sigma =_\beta N[\zeta]^\oplus$. Claim (a) follows from Lemma 4.8 and the fact that $M[\theta_1] \twoheadrightarrow_{\mathrm{Bx}} N[\theta_1]$, because $M \twoheadrightarrow_{\mathrm{Bx}} N$ by assumption. For (b) we must observe that, by definition, if $N_1\{x \backslash N_2\}$ is a subterm of N, then for all $X \in \mathrm{MV}(N_1)$, x does only occur bound in $\theta_2 \,; \zeta$, and thus $\ulcorner M[\theta_1]\!\downarrow_{\mathrm{x}}[(\theta_2 \,; \zeta)\!\downarrow_{\mathrm{x}}]\urcorner = \ulcorner M[\theta_1 \,; \theta_2 \,; \zeta]\!\downarrow_{\mathrm{x}}\urcorner$, where $\ulcorner K \urcorner$ denotes K with all explicit substitutions removed, for arbitrary K. Therefore $t^\sigma =_\beta N[\theta_1 \,; \theta_2 \,; \zeta]^\oplus =_\beta N[\zeta]^\oplus$, and since $s \twoheadrightarrow_\beta t$ by (a), $s^\sigma =_\beta N[\zeta]^\oplus$, as required. $\qquad\square$

The lemma works, because the arguments of the free variables in the term and the leading abstractions in the substitution, take over the role of the explicit substitutions, as can be seen in the following example:

Example 4.11. Let $M = (\lambda x.(\lambda y.Z)\mathsf{b})\mathsf{a}$ be a λx-term, and $\zeta = [Z \mapsto \mathsf{f}(x, y)]$ a grafting. Now, according to Def. 4.9, $\langle s, \sigma \rangle$, where $s = (\lambda x.(\lambda y.z(x, y))\mathsf{b})\mathsf{a}$ and $\sigma = [z \mapsto \lambda xy.\mathsf{f}(x, y)]$ is a λ-extension of $\langle M, \zeta \rangle$, with, $\theta_1 = [Z \mapsto Z(x, y)]$ and $\theta_2 = [\lambda xy.Z]$. We check both cases of Lemma 4.10:

(i) $s^\sigma = (\lambda x.(\lambda y.(\lambda xy.\mathsf{f}(x, y))(x, y))\mathsf{b})\mathsf{a} =_\beta (\lambda x.(\lambda y.\mathsf{f}(x, y))\mathsf{b})\mathsf{a} = M[\zeta]^\oplus$.
(ii) Let $N = Z\{y \backslash \mathsf{b}\}\{x \backslash \mathsf{a}\}$. Then $M \twoheadrightarrow_{\mathrm{x}} N$. Let $t = z(\mathsf{a}, \mathsf{b})$. Now $t^\sigma =_\beta \mathsf{f}(\mathsf{a}, \mathsf{b}) = M[\zeta]^\oplus$. Since $s =_\beta t$, this means that $s^\sigma =_\beta M[\zeta]^\oplus$, as required. (Note that the $\cdot^\oplus$ operation also reduces to x-normal form.)

Translating patterns. Now we define a translation between pairs of prepatterns and presubstitutions, on the one hand, and linear, passive λx-terms and graftings on the other. Because the notion of pattern is parametrized by a sequence of variables, the translation operations are so as well.

Definition 4.12. *Let $\mathbf{P}^+_{\overline{x}}$, a mapping which maps pairs of linear, passive λx-terms containing no explicit substitutions and graftings to pairs of local $\overline{x}$-prepatterns and substitutions, be defined as follows:*

$$\begin{aligned}
\mathbf{P}^+_{\overline{x}}\langle Y, [Y \mapsto M] \rangle &= \langle y(\overline{x}), [y \mapsto \lambda\overline{x}.M^\oplus] \rangle \\
\mathbf{P}^+_{\overline{x}}\langle y(\overline{P_n}), \zeta \rangle &= \langle y(\overline{p_n}), \sigma \rangle && \text{if } \mathbf{P}^+_{\overline{x}}\langle P_i, \zeta{\restriction}P_i \rangle = \langle p_i, \sigma{\restriction}p_i \rangle \\
\mathbf{P}^+_{\overline{x}}\langle f(\overline{P_n}), \zeta \rangle &= \langle f(\overline{p_n}), \sigma \rangle && \text{if } \mathbf{P}^+_{\overline{x}}\langle P_i, \zeta{\restriction}P_i \rangle = \langle p_i, \sigma{\restriction}p_i \rangle \\
\mathbf{P}^+_{\overline{x}}\langle \lambda y.P, \zeta \rangle &= \langle \lambda y.p, \sigma \rangle && \text{if } \mathbf{P}^+_{\overline{x}y}\langle P, \zeta \rangle = \langle p, \zeta_\star \rangle
\end{aligned}$$

where $\zeta{\restriction}P$ denotes the restriction of grafting/substitution ζ to the metavariables/-variables of λx-term/preterm P. With $\mathbf{P}^-_{\overline{x}}$ we denote the inverse of $\mathbf{P}^+_{\overline{x}}$.[2]

---

[2] Like $\cdot^\ominus_{\overline{x}}$, the $\mathbf{P}^-_{\overline{x}}$ operation is not actually a mapping, but this is no problem for the same reason. See footnote 1.

As usual, we generalize the above operations to (local pattern) substitutions and (linear, passive) graftings, in the obvious way. Note that the first element of the result of $\mathbf{P}_{\overline{x}}^-\langle\cdot,\cdot\rangle$ and $\mathbf{P}_{\overline{x}}^+\langle\cdot,\cdot\rangle$ does not depend on the second.

Lemma 4.13. *Let P be a linear, passive λx-term and ζ a grafting, both containing no explicit substitutions. Then $\mathbf{P}_{\overline{x}}^+\langle P,\zeta\rangle$ is a λ-extension of $\langle P,\zeta\rangle$, for arbitrary list of variables $\overline{x}$.*

Proof. By the fact that *all* bound variables in scope are added as arguments to the free variables. $\qquad\square$

Example 4.14. Consider the linear, local λx-terms $P = \mathsf{f}(\lambda xy.\mathsf{g}(Z,x))$ and $Q = \mathsf{map}(\lambda x.Z,\mathsf{nil})$, and the grafting $\zeta = [Z \mapsto \mathsf{f}(x)]$. Then:

$$\mathbf{P}_{\emptyset}^+\langle P,\zeta\rangle = \langle \mathsf{f}(\lambda xy.\mathsf{g}(z(x,y),x)),[z\mapsto \lambda xy.\mathsf{f}(x)]\rangle$$
$$\mathbf{P}_{\emptyset}^+\langle Q,\zeta\rangle = \langle \mathsf{map}(\lambda x.Z(x),\mathsf{nil}),[z\mapsto \lambda x.\mathsf{f}(x)]\rangle$$

4.3   Proof of the Prefix Property

We repeat the text of Theorem 4.1, for convenience:

Theorem 4.1 (Prefix Property). *Let $\mathcal{H}^\omega$ be the ω-labelling of a non-collapsing HRS, s a term, p a local $\overline{x}$-pattern and σ a substitution. If $s \to_{\mathcal{H}^\omega} p^\sigma$, then there exist a local $\overline{x}$-pattern q and a substitution τ, such that $s = q^\tau$, $\mathrm{fam}(p) \geq \mathrm{fam}(q)$, and either:*

- *$q \to_{\mathcal{H}^\omega} p^\upsilon$, for some substitution υ such that $\upsilon\,;\tau = \sigma$; or*   (trm)
- *$q = p$ and $\tau \to_{\mathcal{H}^\omega} \sigma$.*   (sub)

Proof. In the course of this proof we use letters subscripted with a $\star$ (e.g. $s_\star, P_\star, \sigma_\star \ldots$) to denote preterms and presubstitutions. Letters without $\star$-subscript denote terms and substitutions on the term level, or λx-terms and graftings.

We prove the theorem by induction on the context of the step $s \to_{\mathcal{H}^\omega} p^\sigma$. If p has a variable as head, then the (sub) case is trivially satisfied. If the step does not occur at the head, then the result follows easily from the induction hypothesis.

So, assume $s = l^\rho$ and $p^\sigma = r^\rho$, for some rule $\lambda\overline{x}.l \to \lambda\overline{x}.r \in R$ and substitution ρ. We cast everything down to the preterm level, i.e.: $s_\star := s^\flat$, $l_\star := l^\flat$, $\rho_\star := \rho^\flat$, $r_\star := r^\flat$, $p_\star := p^\flat$ and $\sigma_\star := \sigma^\flat$. Furthermore, let $\langle P,\zeta\rangle := \mathbf{P}_{\overline{x}}^-\langle p_\star,\sigma_\star\rangle$. Now, by definition, P is a linear, passive λx-term, and $P[\zeta]^\oplus =_\beta p_\star^{\sigma_\star}$. $P[\zeta]$ is a Bx-normal form and $r_\star^{\rho_\star} =_\beta p_\star^{\sigma_\star} =_\beta P[\zeta]^\oplus$, so $r_\star^{\rho_\star} \twoheadrightarrow_\beta P[\zeta]^\oplus$, and from this and Lemma 4.8 it follows that:
$$R[\mu] \twoheadrightarrow_{\mathrm{Bx}} P[\zeta]$$
where $R := (r_\star)_{\overline{x}}^\ominus$ and $\mu := (\rho_\star)_{\overline{x}}^\ominus$. Now, we can apply the λx-Prefix Property (Theorem 4.6, ii) and obtain a linear, passive λx-term P', with $Sym(P') = Sym(P)$, and graftings η, κ such that:

$$P'[\eta] = R[\mu], \quad P' \twoheadrightarrow_{\mathrm{Bx}} P[\kappa] \quad \text{and} \quad \kappa\,;\eta \twoheadrightarrow_{\mathrm{Bx}} \zeta$$

Because of the first equality above, we can apply Lemma 4.4 and obtain a linear, passive grafting μ' and graftings η', κ with $Sym(\eta') \subseteq Sym(R)$ and $Sym(\mu') \subseteq Sym(P')$, such that:

$$P'[\eta'] = R[\mu'], \quad \eta' \, ; \kappa = \eta \quad \text{and} \quad \mu' \, ; \kappa = \mu$$

Now the only thing we have to do, is translate the obtained λx-terms and graftings to terms and substitutions, using the techniques developped in the previous subsection. This is not hard, in principle, but a cumbersome operation.

Let $\langle \rho'_\star, \tau_\star \rangle := \mathbf{P}^+_\emptyset \langle \mu', \kappa \rangle$. Then $\rho'_\star$ is a local pattern substition, and by Lemma 4.13, $\langle \rho'_\star, \tau_\star \rangle$ is a λ-extension of $\langle \mu', \kappa \rangle$. Because the object variables of $r_\star$ and $\rho'_\star$ are disjoint, it follows that $\langle r_\star^{\rho'_\star}, \tau_\star \rangle$ is a λ-extension of $\langle R[\mu'], \kappa \rangle$ and thus of $\langle P'[\eta'], \kappa \rangle$. Let θ be the grafting such that $\rho'_\star = (\mu' \, ; \theta)^\oplus$ and let $\eta'' := \eta' \, ; \theta$. Next, let $\langle P'_\star, \eta''_\star \rangle := \mathbf{P}^+_{\overline{x}} \langle P', \eta'' \rangle$. By definition, there exists a grafting θ' such that $P'_\star = P'[\theta']^\oplus$, and because $P' \twoheadrightarrow_{Bx} P[\kappa]$ it is the case that $P'_\star = P[\kappa \, ; \theta']^\oplus$. Now, let $\langle P_\star, \kappa_\star \rangle := \mathbf{P}^+_{\overline{x}} \langle P, \kappa \, ; \theta' \rangle$. Since $P_\star$ does not depend on $\kappa \, ; \theta'$, $P_\star = p_\star$. We know now that $P_\star^{\kappa_\star} =_\beta P[\kappa \, ; \theta']^\oplus =_\beta P'[\theta']^\oplus =_\beta P'_\star$.

Finally, we take: $q_\star = l_\star^{\rho'_\star}$, $\upsilon_\star = \kappa_\star \, ; \eta''_\star$. Then:

- $q_\star$ is a local $\overline{x}$-prepattern, because $l_\star$ is a local $\overline{x}$-prepattern and $\rho'_\star$ is a local pattern substitution;
- $r_\star^{\rho'_\star} =_\beta P'^{\eta''_\star}_\star =_\beta P_\star^{\kappa_\star; \eta''_\star} =_\beta p_\star^{\upsilon_\star}$;
- $s_\star = l_\star^{\rho_\star} =_\beta l_\star^{\rho'_\star ; \tau_\star} =_\beta q_\star^{\tau_\star}$;
- $\upsilon_\star \, ; \tau_\star =_\beta \kappa_\star \, ; \eta''_\star \, ; \tau_\star =_\beta \sigma_\star$;
- $\mathrm{fam}(p_\star) = \mathrm{fam}(P') = \mathrm{fam}(P'[\eta']) = \mathrm{fam}(R[\rho']) \geq \mathrm{fam}(l_\circ[\rho']) = \mathrm{fam}(q_\star)$, where the second equality holds because $Sym(\eta') \subseteq Sym(r)$, all labels in r are the same, p and r have at least one symbol in common because r is non-collapsing, and thus $Sym(\eta') \subseteq Sym(p)$.

Now we take $q := q_\star^\sharp$, $\upsilon := \upsilon_\star^\sharp$ and $\tau := \tau_\star$ to satisfy the (trm) case. $\qquad \square$

5   Finite Family Developments

In this section we apply the prefix property of the previous section to prove that all family developments of HRSs are finite. We restrict our attention to non-collapsing HRSs first. In the next section, we will describe a way to generalize the result to collapsing HRSs as well.

Families are formalized by labelling all function symbols with natural numbers, as defined in Def. 3.1. We prove that the resulting system is terminating if we restrict the labels to some finite bound. The proof is inspired by the proof by Van Oostrom [17]. The differences between this proof and the one by Van Oostrom are the following:

- We use a different method of labelling. Our labelling has the property that one step of the labelled HRS corresponds exactly to one step in the original. Also, our notion of labelling is an instance of the abstract notion of labelling put forth in [18,14].

– In Van Oostrom's paper, the proof of Lemma 15 is omitted. Here, we give a proof of that lemma (adapted for the different method of labelling) by reducing it to the Prefix Property.

Lemma 5.1. *Let $\mathcal{H}^\omega$ be the labelling of a non-collapsing HRS, s be a term, p a local pattern, $\ell \in \mathbb{N}$ a label and τ and σ substitutions such that for any $x \in Dom(\sigma)$, $\sigma(x)$ has a function symbol labelled with ℓ as head. If $s^\sigma \twoheadrightarrow_{\mathcal{H}^\omega} p^\tau$, then either:*

– $\mathrm{fam}(p) \geq \ell$; or   (int)
– $s \twoheadrightarrow_{\mathcal{H}} p^\upsilon$, for some υ such that $\upsilon \, ; \sigma \twoheadrightarrow_{\mathcal{H}^\omega} \tau$.   (ext)

Proof. By induction on the length of the reduction $s^\sigma \twoheadrightarrow_{\mathcal{H}^\omega} p^\tau$. If the length is 0, there are two cases: if a substitution υ exists such that $s = p^\upsilon$, then the conditions of the (ext) case are trivially satisfied; otherwise, we show (int) by induction on p, using the assumption on the structure of σ in the base case.

Otherwise, suppose $s^\sigma \twoheadrightarrow_{\mathcal{H}^\omega} s' \to_{\mathcal{H}^\omega} p^\tau$. By Theorem 4.1, there exist a local pattern q and substitution σ' such that $s' = q^{\sigma'}$, $\mathrm{fam}(p) \geq \mathrm{fam}(q)$ and either (trm) $q \twoheadrightarrow_{\mathcal{H}^\omega} p^{\upsilon'}$ and $\upsilon' \, ; \sigma' = \tau$; or (sub) $p = q$ and $\sigma' \twoheadrightarrow_{\mathcal{H}^\omega} \tau$. Applying the induction hypothesis to $s^\sigma \twoheadrightarrow_{\mathcal{H}^\omega} q^{\sigma'}$ yields that one of the following cases must apply:

– *(int)* $\mathrm{fam}(q) \geq \ell$, but then $\mathrm{fam}(p) \geq \ell$ by transitivity of $\geq$.
– *(ext)* $s \twoheadrightarrow_{\mathcal{H}^\omega} q^\upsilon$ and $\upsilon \, ; \sigma \twoheadrightarrow_{\mathcal{H}^\omega} \sigma'$, for some substitution υ. We distinguish the following cases:
  - *(trm)* $s \twoheadrightarrow_{\mathcal{H}^\omega} q^\upsilon \twoheadrightarrow_{\mathcal{H}^\omega} p^{\upsilon';\upsilon}$ and $\upsilon' \, ; \upsilon \, ; \sigma \twoheadrightarrow_{\mathcal{H}^\omega} \upsilon' \, ; \sigma' = \tau$.
  - *(sub)* $s \twoheadrightarrow_{\mathcal{H}^\omega} q^\upsilon = p^\upsilon$ and $\upsilon \, ; \sigma \twoheadrightarrow_{\mathcal{H}^\omega} \sigma' \twoheadrightarrow_{\mathcal{H}^\omega} \tau$.
  In both cases the (ext) case of the lemma is satisfied.   □

Theorem 5.2. *Let $\mathcal{H}^\omega$ be the labelling of a non-collapsing HRS, and let $\mathcal{R}$: $s_1 \to_{\mathcal{H}^\omega} s_2 \to_{\mathcal{H}^\omega} \cdots$ be a $\mathcal{H}^\omega$-reduction. $\mathcal{R}$ is finite, if and only if there is a $\ell_{\max} \in \mathbb{N}$ such that $\mathrm{fam}(s_i) \leq \ell_{\max}$ for all s_i.*

Proof. ($\Rightarrow$): Trivial, because a finite reduction has a finite amount of labels.

($\Leftarrow$): We prove the theorem by showing that $\mathcal{H}^\omega = \langle \Sigma^\omega, R^\omega \rangle$ is terminating if we restrict it to rules $l \to r \in R^\omega$ where $\mathrm{fam}(r) \leq \ell_{\max}$. By the fact that an HRS is terminating if and only if r^σ is terminating for every right-hand side r and terminating substitution σ [17, Lemma 8], it suffices to show termination of $(s^\ell)^\sigma$, for every non-(subst-subst)-collapsing[3], term s, label $\ell \in \mathbb{N}$ and terminating substitution σ. We do this by induction on $(\ell_{\max} - \ell)$.

Let $(s^\ell)^\sigma$ be a minimal non-terminating term. Since it is minimal, we can assume that this infinite reduction contains at least one head step, and $s = a(s_1, \ldots, s_n)$, so $s^\ell = a'((s_1^\ell)^\sigma, \ldots, (s_n^\ell)^\sigma)$. By minimality, the s_i^ℓ are terminating. We distinguish the following cases:

---

[3] We drop the (context-subst) condition of Def 2.5, because subterms of non (context-subst)-collapsing terms can be (constext-subst)-collapsing, meaning that an infinite reduction from a minimal counter example might not contain a head step.

- If a is the function symbol f, then $a' = f^{\ell}$. Since the first head step strictly increases the label, termination follows from the induction hypothesis.
- If a is a variable, then it must be in the domain of σ (otherwise, a head step would not be possible, contradicting minimality). Suppose:

$$\sigma(a) = \lambda \overline{x}.b(t_1, \ldots, t_m)$$

and thus $(s^{\ell})^{\sigma} = b(t_1^{\sigma'}, \ldots, t_m^{\sigma'})$, where $\sigma' = [x_1 \mapsto s_1^{\ell}, \ldots, x_n \mapsto s_n^{\ell}]$. We prove the lemma by a nested induction on the order $\ll_{\mathcal{H}^{\omega}}$ starting from $\sigma(a)$. We distinguish the following cases:

- Suppose $b = f^{\ell'}$. Then an infinite reduction from $(s^{\ell})^{\sigma}$ looks like:

$$f^{\ell'}(t_1^{\sigma'}, \ldots, t_m^{\sigma'}) \twoheadrightarrow_{\mathcal{H}^{\omega}} f^{\ell'}(t_1, \ldots, t_m) = l^{\tau} \rightarrow_{\mathcal{H}^{\omega}} (r^{\ell''})^{\tau} \twoheadrightarrow_{\mathcal{H}^{\omega}} \cdots$$

  where $l \rightarrow r^{\ell'} \in R^{\omega}$. Since a is a variable and s is not (subst-subst)-collapsing, we know that the s_i^{ℓ} have function symbols labelled with ℓ as head, and we can apply Lemma 5.1 to l^{τ}:
    * (int): $\mathrm{fam}(l) \geq \ell$: It follows by the fact that, by construction, $\ell'' = \mathrm{fam}(l)+1$, that $\ell'' \geq \ell$. Thus, by the outermost induction hypothesis, $(r^{\ell''})^{\tau}$ is terminating, contradicting the assumption that s is not.
    * (ext): $a^{\sigma} \twoheadrightarrow_{\mathcal{H}^{\omega}} l^{\upsilon}$ and $\upsilon \, ; \sigma' \twoheadrightarrow_{\mathcal{H}^{\omega}} \tau$: We know that:

$$a^{\sigma} \twoheadrightarrow_{\mathcal{H}^{\omega}} l^{\upsilon} \rightarrow_{\mathcal{H}^{\omega}} (r^{\ell''})^{\upsilon} \quad \text{and} \quad (r^{\ell''})^{(\upsilon;\sigma)} \twoheadrightarrow_{\mathcal{H}^{\omega}} (r^{\ell''})^{\tau}$$

    Since the left reduction consists of at least one step (the last one), $(r^{\ell''})^{\upsilon}$ is terminating by the nested induction hypothesis. By the right sequence and the fact that σ' is terinating, this yields termination of $(r^{\ell''})^{\tau}$, contradicting the assumption that s is not..
- Suppose b is a variable, it must be in the domain of σ. Suppose $b = x_i$, and $s_i = \lambda \overline{y}.c(u_1, \ldots, u_l)$. Then $(s^{\ell})^{\sigma} = c(u_1, \ldots, u_l)^{(\sigma;\tau)}$ where $\tau = \bigcup_{1 \leq i \leq m}[y_i \mapsto t_i^{\sigma'}]$. Now σ is terminating by assumption, and the $t_i^{\sigma'}$ are terminating by the nested induction hypothesis, so $\sigma \, ; \tau$ is a terminating substitution. Since $c(u_1, \ldots, u_l)$ is a proper subterm of s, termination of $c(u_1, \ldots, u_l)^{(\sigma;\tau)}$ follows by minimality. $\qquad\square$

6 Dealing with Collapsing HRSs

In the previous sections we restricted our attention to non-collapsing HRSs. Both the Prefix Property and FFD do not hold for collapsing HRSs, as is witnessed by the following two counterexamples:

Example 6.1 (Prefix Property). Consider the collapsing HRS $\mathcal{M}u$:

$$\mathsf{mu}(\lambda x.z(x)) \rightarrow z(\mathsf{mu}(\lambda x.z(x))$$

and the following $\mathcal{M}u^{\omega}$-step:

$$\mathsf{mu}^3(\lambda x.\mathsf{f}^2(x)) \rightarrow_{\mathcal{M}u^{\omega}} \mathsf{f}^2(\mathsf{mu}^4(\lambda x.\mathsf{f}^2(x)))$$

It is easy to check that the prefix $p = \mathsf{f}^2(u)$ of the target of the step has no ancestor q that satisfies the requirements of the Prefix Property (Theorem 4.1).

18

Example 6.2 (FFD). Consider the collapsing HRS $\mathcal{L}am$:

$$\mathsf{app}(\mathsf{lam}(\lambda x.z(x), y)) \to z(y)$$

Then one $\mathcal{L}am^\omega$-step is the following:

$$\mathsf{app}^1(\mathsf{lam}^1(\lambda x.\mathsf{app}^1(x, x)), \mathsf{lam}^1(\lambda x.\mathsf{app}^1(x, x)))$$
$$\to_{\mathcal{L}am^\omega} \mathsf{app}^1(\mathsf{lam}^1(\lambda x.\mathsf{app}^1(x, x)), \mathsf{lam}^1(\lambda x.\mathsf{app}^1(x, x)))$$

So we see that $\mathcal{L}am^\omega$ has a one-step cycle, and thus an infinite reduction with bounded labels.

The problem in both cases is that, because of applying a collapsing rule, a function symbol can be directly connected to a previously unconnected function symbol from the context or substitution, or to the root of the term, without the rule leaving any trace in between, in the form of a label. This can be remedied by including "empty" function symbols, named ϵ_α, for each base type α, in the right-hand sides of rules, and "saturating" the left-hand sides of rules with those empty function symbols. The same approach is taken for the first-order case in [14, Chap. 8].

Definition 6.3 (ϵ-lifting).

(i) *The ϵ-lifting Σ^ϵ of a signature Σ, consists of all function symbols of Σ, and, for each base type α, a function symbol $\epsilon_\alpha : \alpha \to \alpha$.*

(ii) *The ϵ-lifting of a term s of type α, written s^ϵ, is defined as follows:*

$$x(s_1, \ldots, s_n)^\epsilon = \epsilon_\alpha(x(s_1^\epsilon, \ldots, s_n^\epsilon)))$$
$$f(s_1, \ldots, s_n)^\epsilon = f(s_1^\epsilon, \ldots, s_n^\epsilon)$$
$$(\lambda x.s_0)^\epsilon = \lambda x.s_0^\epsilon$$

(iii) *The projection operator $|\cdot|_\epsilon$ maps Σ^ϵ-terms to Σ-terms by removing all ϵ_α-symbols.*

(iv) *The set of ϵ-saturations of a pattern p, denoted by $\mathrm{Sat}^\epsilon(s)$, is defined as $\mathrm{Sat}^\epsilon(p) = \mathrm{Sat}^\epsilon_{\mathrm{out}}(p)$, where:*

$$\mathrm{Sat}^\epsilon_{\mathrm{out}}(x(\overline{y_n})) = \{x(\overline{y_n})\}$$
$$\mathrm{Sat}^\epsilon_{\mathrm{out}}(f(\overline{p_n})) = \{f(\overline{q_n}) \mid q_i \in \mathrm{Sat}^\epsilon_{\mathrm{in}}(p_i)\}$$
$$\mathrm{Sat}^\epsilon_{\mathrm{out}}(\lambda x.p_0) = \{\lambda x.q_0 \mid q_0 \in \mathrm{Sat}^\epsilon_{\mathrm{in}}(p_0)\}$$
$$\mathrm{Sat}^\epsilon_{\mathrm{in}}(x(\overline{y_n})) = \{x(\overline{y_n})\}$$
$$\mathrm{Sat}^\epsilon_{\mathrm{in}}(f(\overline{p_n})) = \{\epsilon_\alpha^m(f(\overline{q_n})) \mid m \in \mathbb{N}, q_i \in \mathrm{Sat}^\epsilon_{\mathrm{in}}(p_i)\}$$
$$\mathrm{Sat}^\epsilon_{\mathrm{in}}(\lambda x.p_0) = \{\lambda x.q_0 \mid q_0 \in \mathrm{Sat}^\epsilon_{\mathrm{in}}(p_0)\}$$

where α is the type of p, and $\epsilon_\alpha^m(p)$ is inductively defined by $\epsilon_\alpha^0(p) = p$, $\epsilon_\alpha^{m+1}(p) = \epsilon_\alpha(\epsilon_\alpha^m(p))$.

(v) *The ϵ-lifting of an HRS $\mathcal{H} = \langle \Sigma, R \rangle$ is defined as $\mathcal{H}^\epsilon = \langle \Sigma^\epsilon, R^\epsilon \rangle$, where*

$$R^\epsilon = \bigcup_{l \to r \in R} \{l' \to r^\epsilon \mid l' \in \mathrm{Sat}^\epsilon(l)\}$$

19

The ϵ-liftings of the two counter examples introduced earlier in this section do not have the same problems as their originals:

Example 6.4. The ϵ-lifting of $\mathcal{M}u$ (types of ϵ's omitted):

$$\mathsf{mu}(\lambda x.z(x)) \rightarrow \epsilon(z(\epsilon(\mathsf{mu}(\lambda x.\epsilon(z(\epsilon(x))))))) $$

A $(\mathcal{M}u^\epsilon)^\omega$ step corresponding to the step of Ex. 6.1 is:

$$\mathsf{mu}^3(\lambda x.\mathsf{f}^2(x)) \rightarrow_{(\mathcal{M}u^\epsilon)^\omega} \epsilon^4(\mathsf{f}^2(\epsilon^4(\mathsf{mu}^4(\lambda x.\epsilon(\mathsf{f}^2(\epsilon(x)))))))$$

Take the corresponging prefix $p = \epsilon^4(\mathsf{f}^2(y))$. Now, the Prefix Property is satisfied with $q = \mathsf{mu}^3(\lambda x.\mathsf{f}^2(x))$, $\tau = \emptyset$ and $\upsilon = [z \mapsto \epsilon^4(\mathsf{mu}^4(\lambda x.\epsilon(\mathsf{f}^2(\epsilon(x)))))]$.

Example 6.5. The ϵ-lifting of $\mathcal{L}am$ consists of (among others) the following rules:

$$\mathsf{app}(\mathsf{lam}(\lambda x.z(x), y)) \rightarrow \epsilon(z(\epsilon(y)))$$
$$\mathsf{app}(\epsilon(\mathsf{lam}(\lambda x.z(x))), y) \rightarrow \epsilon(z(\epsilon(y)))$$
$$\mathsf{app}(\epsilon(\epsilon(\mathsf{lam}(\lambda x.z(x)))), y) \rightarrow \epsilon(z(\epsilon(y)))$$

Then a $(\mathcal{L}am^\epsilon)^\omega$-step corresponding to the step of Ex. 6.2 is the following:

$$\mathsf{app}^1(\mathsf{lam}^1(\lambda x.\mathsf{app}^1(x,x)), \mathsf{lam}^1(\lambda x.\mathsf{app}^1(x,x)))$$
$$\rightarrow_{(\mathcal{L}am^\epsilon)^\omega} \epsilon^2(\mathsf{app}^1(\epsilon^2(\mathsf{lam}^1(\lambda x.\mathsf{app}^1(x,x))), \epsilon^2(\mathsf{lam}^1(\lambda x.\mathsf{app}^1(x,x)))))$$

Now, all redex patterns have a maximum label of 2, instead of 1.

We can show that ϵ-lifting does not interfer with rewriting:

Lemma 6.6. *Let $\mathcal{H}$ be an HRS and $\mathcal{H}^\epsilon$ its ϵ-lifting. Then:*

(i) *Let s,t be $\mathcal{H}$-terms such that $s \rightarrow_\mathcal{H} t$. For each $\mathcal{H}^\epsilon$-term s' such that $|s'|_\epsilon = s$ there exists a unique $\mathcal{H}^\epsilon$-term t' such that $|t'|_\epsilon = t$ and $s' \rightarrow_{\mathcal{H}^\epsilon} t'$.*
(ii) *Let s,t be $\mathcal{H}^\epsilon$-terms. If $s \rightarrow_{\mathcal{H}^\epsilon} t$, then $|s|_\epsilon \rightarrow_\mathcal{H} |t|_\epsilon$.*

Proof. (i) is proved by induction on the step, noting that the type of the ϵ's prohibits them from interferring with β-equivalence, and (ii) follows from the facts that, for each s, there is an $s' \in \mathrm{Sat}^\epsilon(s)$ such that s^ϵ matches a subterm of s', and that, for any term t, if $t \in \mathrm{Sat}^\epsilon(s)$, then $|t|_\epsilon = s$.

Proposition 6.7. *For any HRS $\mathcal{H}$, its ϵ-lifting $\mathcal{H}^\epsilon$ is a non- collapsing HRS.*

Proof. It is easy to see, that for all collapsing terms s, s^ϵ is non-collapsing. $\square$

Theorem 6.8 (FFD). *Let $(\mathcal{H}^\epsilon)^\omega$ be the $\epsilon\omega$-labelling of an HRS, and let $\mathcal{R}$: $s_1 \rightarrow_{(\mathcal{H}^\epsilon)^\omega} s_2 \rightarrow_{(\mathcal{H}^\epsilon)^\omega} \cdots$ be a $(\mathcal{H}^\epsilon)^\omega$-reduction. $\mathcal{R}$ is finite, if and only if there is a $\ell_{\max} \in \mathbb{N}$ such that $\mathrm{fam}(s_i) \leq \ell_{\max}$ for all s_i.*

Proof. By Prop. 6.7 and Theorem 5.2. $\square$

7 Applications and Further Research

The Prefix Property and Finite Family Developments are useful tools for proving various properties of HRSs. For example, an alternative proof of termination of the simply typed λ-calculus (encoded as an HRS) uses FFD. Also, in a work in progress by the author, FFD is used to prove the termination of a higher-order standardization procedure. This result can be used to formalize the notion of equivalence of reductions, in a similar way as is done in [14].

For future research, it might be interesting to further investigate the relation between FFD and the Dependency Pair method [1], one of the most powerful (first-order) termination techniques of today, both in the higher-order and first-order case. Since FFD and the Dependency Pair method both essentially depend on the same principle, that an infinite reduction must have an unbounded creation depth, it the author's conjecture that FFD, or the Prefix Property, can be used to design a higher-order Dependency Pair method.

A Termination of the Simply Typed λ-Calculus

The most well-known proof of termination of the simply typed λ-calculus is the proof using strong computability due to Tait [13]. Here we present a termination proof of an HRS which encodes the simply typed λ-calculus, using the Finite Family Developments result of the present paper.[4]

The HRS we consider has an infinite number of rules. Let α, β range over (codes of) types, and let the following signature be given:

$$\mathsf{T}_\alpha : term \to term$$
$$\mathsf{app} : term \to term \to term$$
$$\mathsf{lam} : (term \to term) \to term$$

The simply typed λ-calculus can be encoded as the following HRS, $\mathcal{L}am^\to$:

$$\mathsf{app}(\mathsf{T}^n_{\alpha\to\beta}(\mathsf{lam}(\lambda x.Z(x))), Y) \to \mathsf{T}_\beta(Z(\mathsf{T}_\alpha(Y)))$$

for every $n > 0$, where $f^n(s)$ is defined (here) as: $f^0(s) = s$ and $f^{n+1}(s) = f(f^n(s))$. Termination of $\mathcal{L}am^\to$ is proved in Prop. A.3. But first, we show that the HRS encodes the simply typed λ-calculus in the following sense: every reduction in the simply typed λ-calculus can be lifted to a $\mathcal{L}am^\to$-reduction. For this, we use the following map h, which maps each simply typed λ-term to a set of possible $\mathcal{L}am^\to$-encodings of it:

$$\mathrm{h}(x_\alpha) = \{\mathsf{T}^n_\alpha(x) \mid n > 0\}$$
$$\mathrm{h}(f_\alpha) = \{\mathsf{T}^n_\alpha(f) \mid n > 0\}$$
$$\mathrm{h}(\lambda x_\alpha.M_\beta) = \{\mathsf{T}^n_{\alpha\to\beta}(\mathsf{lam}(\lambda x.s)) \mid s \in \mathrm{h}(M), n > 0\}$$
$$\mathrm{h}(M_{\alpha\to\beta}N_\alpha) = \{\mathsf{T}^n_\beta(\mathsf{app}(s,t)) \mid s \in \mathrm{h}(M), t \in \mathrm{h}(N), n > 0\}$$

---

[4] It is noted in [7, p. 31] that termination of simply typed λ-calculus follows from termination of the Hyland–Wadsworth-labelling, a variation of which we use to formalize FFD.

It follows directly from the definition that $h(M) \neq \emptyset$, for all simply typed λ-terms M. Additionally, we show that the map h commutes over substitution:

Lemma A.1. $h(M[x \mapsto N]) = h(M)[x \mapsto h(N)]$.

Proof. By induction on M.

We have the following correspondence between the simply typed λ-calculus and $\mathcal{L}am^{\rightarrow}$:

Lemma A.2. *If* $M \rightarrow_\beta N$ *and* $s \in h(M)$, *then there is a* $t \in h(N)$ *such that* $s \rightarrow_{\mathcal{L}am^{\rightarrow}} t$.

Proof. By induction on the derivation of $M \rightarrow_\beta N$. The interesting case is if the β-step occurs at the head, i.e.:

$$M = (\lambda x_\alpha.M_1)M_2 \quad \text{and} \quad N = M_1[x \mapsto M_2]$$

By definition, this means that $s = \mathsf{T}_\beta^n(\mathsf{app}(\mathsf{T}_{\alpha \rightarrow \beta}^m(\mathsf{lam}(\lambda x.t_1)), t_2))$, where $t_i \in h(M_i)$, for $i \in \{1, 2\}$, and $m, n > 0$. Now:

$$s \rightarrow \mathsf{T}_\beta^{n+1}(t_1[x \mapsto \mathsf{T}_\alpha(t_2)]) \in h(N)$$

where the last inclusion follows from Lemma A.1 and the facts that $\mathsf{T}_\beta^{n+1} \in h(M_1)$ because $t_1 \in h(M_1)$, and $\mathsf{T}_\alpha(t_2) \in h(M_2)$ because $t_2 \in h(M_2)$.

Termination of $\mathcal{L}am^{\rightarrow}$ cannot be proved by any higher-order termination technique that I know of. However, it does follow from Finite Family Developments:

Proposition A.3. $\mathcal{L}am^{\rightarrow}$ *is terminating.*

Proof. Note that $\mathcal{L}am^{\rightarrow}$ is non-collapsing, and consider the ω-labelling of $\mathcal{L}am^{\rightarrow}$, which consists of rules of the form:

$$\mathsf{app}^\ell(((\mathsf{T}_{\alpha \rightarrow \beta}^k)^n(\mathsf{lam}^j(\lambda x.Z(x))), Y) \rightarrow \mathsf{T}_\beta^p(Z(\mathsf{T}_\alpha^p(Y)))$$

where ℓ, j are labels representing the creation depth of the symbols, $\overline{k}$ is a sequence of n such labels and $p = \max(\ell, \overline{k}, j)$. The symbols of the form T_α will be called *type symbols*.

Let the *height* of a type be defined as follows: $ht(a) = 1$ (where a is a base type); $ht(\alpha \rightarrow \beta) = \max(ht(\alpha), ht(\beta)) + 1$. Now we can define the *value* of a type symbol as $Val(\mathsf{T}_\alpha^\ell) = \ell + ht(\alpha)$ and the value of a term as $Val(s) = \max\{Val(f) \mid f \in Sym(s)\}$. Now, a simple induction on the context of the step reviels that, if $s \rightarrow_{\mathcal{L}am^{\rightarrow}} t$, then $Val(s) \geq Val(t)$.

Let $\mathcal{R} : s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \cdots$ be an arbitrary $\mathcal{L}am^{\rightarrow}$-reduction. Because $Val(s_i) \geq Val(s_j)$ if $i < j$, the value of every term in $\mathcal{R}$ is less or equal to $Val(s_0)$, and thus every symbol occurring in the reduction has a creation depth less than or equal to $Val(s_0)$. Therefore, by Finite Family Developments (Theorem 5.2), $\mathcal{R}$ is finite.

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