Recognisability Equals Definability for Finitely Representable Matroids of Bounded Path-Width

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***Abstract*—Let $\mathbb{F}$ be a finite field. We prove that there is an MSO-transduction which, given an $\mathbb{F}$-representable matroid of path-width k, produces a branch-decomposition of width at most $f(k)$, for some function f. As a corollary, any recognizable property of $\mathbb{F}$-representable matroids with bounded path-width is definable in MSO logic, and therefore recognizability is equivalent to MSO-definability on classes of $\mathbb{F}$-representable matroids of bounded path-width. This generalizes the result of Bojańczyk, Grohe and Pilipczuk [*Logical Methods in Computer Science* 17(1), 2021] which asserts the equivalence of the two notions on graphs of bounded linear clique-width.**

***Index Terms*—recognisability, definability, monadic second-order logic, decomposition-width, transduction, matroid, branch-width, path-width**

I. Introduction

A tree-like decomposition, which is obtained from recursive separations of a graph, has been widely used in various fields of computer science such as algorithms design, databases, and logic. Different notions of separations lead to different decompositions like tree-decomposition, branch-decomposition, rank-decomposition and carving-decomposition, to name a few. Such a decomposition is especially handy for designing efficient algorithms. When the *complexity* of separations is bounded, the information that can be communicated across a separation is limited and this allows an efficient processing of the entire graph in a bottom-up manner following the tree structure.

A prominent example of algorithms utilizing tree-like decompositions is Courcelle's theorem [1]. This theorem states that if a graph property Π is *definable* in monadic second-order logic of the second kind with modular counting predicates (CMSO_2) , then the property is *recognizable* in the sense that for each k, there is a tree automaton that can recognise the encodings of tree-decompositions of width at most k of graphs in Π. Courcelle [1] conjectured that the converse statement holds. This conjecture was later proven by Bojańczyk and Pilipczuk [2], [3], establishing that for graphs of bounded tree-width, definability in CMSO_2 is equivalent to recognizability.

It is well known that graphs of bounded tree-width have bounded clique-width, and MSO_1 logic is not weaker than MSO_2 logic on graphs of bounded tree-width. Courcelle, Makowsky, and Rotics [4] extended Courcelle's theorem for tree-width to fixed-parameter tractability of CMSO_1-model checking on graphs of bounded clique-width. The question of whether CMSO_1-definability and recognizability are equivalent on graphs of bounded clique-width was partially answered by Bojańczyk, Grohe, and Pilipczuk [5], who showed that the equivalence holds on graphs of bounded linear clique-width.

A common generalization of tree-width and clique-width of a graph is the branch-width of a binary matroid. For any graph G, one can associate a *graphic* matroid M_G whose bases are precisely the edge sets of G forming a maximal acyclic subgraph of G and M_G is representable over the binary field $\mathbb{F}_2$. It is known that the branch-width of M_G equals the branch-width of G minus one when G is bridgeless [6]. As the tree-width and the branch-width of a graph is bounded by a constant factor of each other, branch-width of a binary matroid generalises tree-width of a graph. On the other hand, clique-width is functionally equivalent to rank-width, that is, $\mathsf{rw}(G) \leq \mathsf{cw}(G) \leq 2^{\mathsf{rw}(G)+1} - 1$ [7], and the rank-width of a graph can be associated with the branch-width of a binary matroid in the following way. For a graph G on the vertex set $\{1, \ldots, n\}$, we define a so-called partitioned matroid M: the ground set of M consists of $2n$ elements $\{e_1, \ldots, e_n\} \cup \{v_1, \ldots, v_n\}$, where e_i's are the standard basis of $\mathbb{F}_2^n$, and each v_i is the vector $\sum_{j \in N(i)} e_j$. Additionally, the ground set of M is equipped with the partition $\{\{e_i, v_i\} : i \in [n]\}$, and the branch-width of M is the width of a branch-decomposition of M which assigns the parts of the partition to the leaves, and not the individual elements of the ground set. It can be easily shown that the branch-width of M equals twice the rank-width of G [8].

Hliněný [9] proved a generalization of [1] and [4] for matroids representable over any finite field $\mathbb{F}$: if a class Π of $\mathbb{F}$-representable matroids is CMSO-definable on $\mathbb{F}$-representable matroids of bounded branch-width, then the encodings of branch-decompositions of bounded width of matroids in Π are recognised by a tree automaton. It is thus natural to conjecture that on $\mathbb{F}$-representable matroids of bounded branch-width, $\mathbb{F}$ finite, definability in CMSO logic on matroids is equivalent to recognizability.

The main result of this paper is that recognizability and CMSO-definability are equivalent on $\mathbb{F}$-representable matroids of bounded path-width, for a finite field $\mathbb{F}$, thus partially establishing the conjecture (Theorem III.1). At the heart of our proof lies an MSO-transduction τ_k, for each integer $k \geq 0$, which nondeterministically produces encodings of some branch-decompositions of width at most $f(k)$, from a $\mathbb{F}$-representable matroid of path-width k given by its *independence representation* (Theorem III.2). Once such an MSO-transduction τ_k is demonstrated, a standard argument using the *Backwards Translation Theorem* [10] then implies the claimed result: if encodings of branch-decompositions of width at most $f(k)$ of a class $\mathscr{L}$ can be recognised by a tree automaton, then these encodings form a CMSO-definable class [1, Proposition 5.4]. Then the Backwards Translation Theorem implies that the matroids on which τ_k produces such encodings of branch-decompositions of width at most $f(k)$ is CMSO-definable as well. On the class of $\mathbb{F}$-representable matroids of path-width at most k, these matroids coincide with $\mathscr{L}$.

In fact, we prove the conjecture for a more general class of matroids than the class of matroids representable over a finite field $\mathbb{F}$; the condition of $\mathbb{F}$-representability in the previous paragraph can be replaced by the so-called *strongly-pigeonhole property*. This property relies on an alternative width, namely on *decomposition-width* (see Subsection II-A), which was independently introduced by Král' [11] and Strozecki [12] in order to extend Hliněný's theorem [9]. The branch-width is bounded by the decomposition-width but the decomposition-width is not necessarily bounded by a function of the branch-width. When a class of matroids has the property that the two measures are functionally equivalent, we say that the class is *strongly-pigeonhole*. This is the case for the class of matroids representable over a finite field. It was shown in [13] that, besides $\mathbb{F}$-representable matroids for finite $\mathbb{F}$, there are other classes of matroids which are strongly-pigeonhole such as the fundamental transversal matroids, lattice path matroids, 3-connected bicircular matroids and 3-connected $\mathbb{H}$-gain graphic matroids with a finite group $\mathbb{H}$.

To obtain our main result, we consider an algebra on so-called *k-ported set systems* to rewrite a matroid of bounded path-width (Subsection II-E). For a strongly-pigeonhole class of matroids of bounded path-width, such a rewriting system of a matroid allows us to view the linearised decomposition as a word over a finite alphabet. Then, we follow the approaches taken by the previous works [5], [2]. Using Simon's Factorisation Forest Theorem, the word can be recursively factored into subwords depicted by a bounded depth factorisation tree. The construction of the MSO-transduction from strongly-pigeonhole matroids of bounded path-width to encodings of a decomposition of bounded width is done inductively based on the depth of the factorisation tree.

A key technical issue encountered when following this approach is that, while in the binary case we can guess the bipartition into two k-ported set systems, we cannot guess the unranked factorisation (see Section IV for the definition of the factorisation). Instead of guessing the unranked factorisation, we prove the existence of an equivalence relation, which can be used to partition the elements of the unranked factorisation and which has the property that we can guess the unranked factorisation when restricted to each equivalence class through a suitable colouring of the elements in the ground set. However, we can prove the existence of this colouring only in some special case of unranked factorisation (called *without trivial hyperedges*). In order to deal with the other cases, we show that one can modify the rewriting with k-ported set systems and obtain another one without trivial hyperedges, and we strongly rely on the fact that the rewriting with k-ported set systems is a factor of a word defining a matroid. It is worth mentioning that the other places where we use the fact that the input is a matroid is for showing that the rooted layout computed by the MSO-transduction has small width.

Related works: The extension of the well-established connection between MSO, Automata, and algebra from graphs to representable matroids has already been investigated in [14], in which the author also unify the results of this kind under category theory. It is known that, if given by the *null representation*, the case of $\mathbb{F}$-representable matroids of bounded branch-width reduces to the case of graphs [14, Theorem 6.2] (see also [15], [16]). In our present work, we deal with matroids described by the *independence representation*. This standard representation is *a priori* significantly less descriptive than the null representation. Indeed, in the former, a representation is fixed and given as part of the input, while the latter only describes which ground subset is an independent. Clearly, the independence representation can be obtained within MSO from the null representation. However, whether an MSO-transduction can produce the null representation of an $\mathbb{F}$-representable matroid given by its independence representation is unknown in the general case, and we conjecture a negative answer to this question.[1] Hence, our result is not implied by the results from [14]. Furthermore, even if our above conjecture turns out to be wrong, using the null representation prevents the generalization to classes of non-representable matroids for which MSO is still tractable, *e.g.*, the previously-mentioned examples from [13]. In contrast, our technique does not rely on the representation, and do work for strongly-pigeonhole matroids which are not representable.

Outline: In Section II we present necessary terminologies. In Section III we state our main results, namely Theorems III.1 and III.2, and we prove the former assuming the latter. The proof of Theorem III.2 is presented in Section IV. Due to space constraints, most of the proofs are omitted.

[1] By tuning our construction, we could actually obtain such a transduction for representable matroids of bounded linear-branch-width.

II. Preliminaries

The set of positive integers (including 0) is denoted by $\mathbb{N}$ and for a positive integer n, the set $\{1, \ldots, n\}$ of integers is denoted as $[n]$. For $m, n \in \mathbb{N}$, we write $[\![m, n]\!]$ for the interval $\{m, \ldots, n\}$. For a set V and $x \in V$, the singleton $\{x\}$ shall be often written simply as x. The power set of a finite set V is denoted by 2^V, the complement of a subset X of V is denoted by $V \setminus X$ (or $\overline{X}$ for short when V is clear from the context) and we write $|V|$ to denote the size of V. For an equivalence relation $\equiv$ on $V \times V$, we denote by $V/\!\equiv$ the set of equivalence classes of $\equiv$ and we write $[Y]_{\equiv}$ to denote the equivalence class of $Y \in V$. Recall that the set of equivalence classes forms a partition.

A *set system* $\mathbf{S}$ is a pair $(S, \mathcal{S})$ where S is a finite set and $\mathcal{S}$ is a collection of subsets of S. We refer to S as the *ground set* and to members of $\mathcal{S}$ as *hyperedges*. We use boldface capital letters to denote set systems, *e.g.*, $\mathbf{S}$, $\mathbf{M}$; capital letters for ground sets, *e.g.*, S, M; and calligraphic letters for set of hyperedges, *e.g.*, $\mathcal{S}$, $\mathcal{M}$. We follow [17] for our graph terminology. For a graph G, we denote by $V(\mathsf{G})$ its vertex set, and by $E(\mathsf{G})$ its edge set; an edge between x and y in an undirected graph is denoted by xy (equivalently yx). It is common to call vertices of a tree *nodes*. A *caterpillar* is a tree where the remaining graph after removing the leaves is a path. A *rooted tree* is a tree T with a distinguished node called the *root* and denoted $\mathsf{root}(\mathsf{T})$.

A. Decomposition width

Let $\mathbf{S}$ be a set system, and U a subset of S. Two subsets X and Y of U are *U-equivalent*, denoted by $X \equiv_{\mathbf{S}}^{U} Y$, when, for all $Z \subseteq \overline{U}$, the set $X \cup Z \in \mathcal{S}$ if and only if $Y \cup Z \in \mathcal{S}$. Note that $\equiv_{\mathbf{S}}^{U}$ is an equivalence relation on subsets of U. Intuitively, subsets of U that are $\equiv_{\mathbf{S}}^{U}$-equivalent are indistinguishable by any subset of $\overline{U}$. Geometrically, we may think of the $\equiv_{\mathbf{S}}^{U}$-equivalence classes as the possible "shadows" of subsets of U when viewed from $\overline{U}$. Computationally, for a subset Z of E, we can think of the $\equiv_{\mathbf{S}}^{U}$-equivalence class that contains $Z \cap U$ as the partial evaluations of membership in $\mathcal{S}$ when we have only looked at U. We let $\mathsf{shadow}_{\mathbf{S}}(U)$ be the number of equivalence classes of $\equiv_{\mathbf{S}}^{U}$ for the set system $\mathbf{S}$.

If S is a finite set, a *rooted layout of S* is a pair (T, δ) formed by a rooted tree T and a bijection $\delta : S \to L_{\mathsf{T}}$ between S and the set L_{T} of leaves[2] of T. For a rooted layout (T, δ) and a node v of T, we denote by $S_v \subseteq S$ the set of elements s such that v is on the unique path between $\mathsf{root}(\mathsf{T})$ and $\delta(s)$. A *rooted linear layout* is a rooted layout (T, δ) where T is a caterpillar and $\mathsf{root}(\mathsf{T})$ is an internal node which is adjacent to at most one node.

Definition II.1. *Let $f : 2^S \to \mathbb{N}$ be a set function with $f(\emptyset) = 0$, and (T, δ) a rooted (linear) layout of S. The f-width of a node v of T is defined as*

$$\max\{f(\bigcup_{w \in F} S_w) \mid F \text{ is a subset of children of } v\},$$

and the f-width of (T, δ) is defined as the maximum f-width over all nodes of T. The (linear) f-width of S is defined as the minimum f-width over all rooted (linear) layouts (T, δ) of S. The (linear) decompostion-width of a set system $\mathbf{S}$ is the (linear) $\mathsf{shadow}_{\mathbf{S}}$-width of S, denoted by $\mathsf{dw}(\mathbf{S})$ ($\mathsf{ldw}(\mathbf{S})$).

It is worth mentioning that our definition of rooted layout is, contrary to the usual definition in the literature, not binary, but as one checks easily, any rooted layout can be binarised in the usual way by taking any linear ordering of the children of a node with more than two children and still keeps the same width. As in [5], the use of unranked rooted trees is important, otherwise we won't be able to transduce a decomposition since linear orderings cannot be defined in MSO logic.

B. Matroids

We follow [18] for our matroid terminology. Consider a set system $\mathbf{M} = (M, \mathcal{M})$ where the following are satisfied:

- C1. $\emptyset \notin \mathcal{M}$.
- C2. If $C \in \mathcal{M}$ and $X \subsetneq C$, then $X \notin \mathcal{M}$.
- C3. If C_1 and C_2 are distinct members of $\mathcal{M}$ and there is $e \in C_1 \cap C_2$, then there is some $C_3 \in \mathcal{M}$ where $C_3 \subseteq (C_1 \cup C_2) - e$.

When this is the case, we say that $\mathbf{M}$ is a *matroid circuit-description* and call it simply a *matroid*.[3] For a matroid circuit-description $\mathbf{M}$, the *rank function of* $\mathbf{M}$ is the function $\mathsf{r}_{\mathbf{M}} : 2^M \to \mathbb{N}$ where, for $X \subseteq M$, we have

$$\mathsf{r}_{\mathbf{M}}(X) = \max\{|I| \mid I \subseteq X, \forall Y \subseteq I, Y \notin \mathcal{M}\}$$

(When $I \subseteq M$ is such that none of its subsets belongs to $\mathcal{M}$, I is called an *independent*.) The *local connectivity function of* $\mathbf{M}$ is $\sqcap_{\mathbf{M}} : 2^M \times 2^M \to \mathbb{N}$ where for $X, Y \subseteq M$, $\sqcap_{\mathbf{M}}(X, Y) = \mathsf{r}_{\mathbf{M}}(X) + \mathsf{r}_{\mathbf{M}}(Y) - \mathsf{r}_{\mathbf{M}}(X \cup Y)$. The *connectivity function of* $\mathbf{M}$ is $\lambda_{\mathbf{M}} : 2^M \to \mathbb{N}$ given by $\lambda_{\mathbf{M}}(X) = \sqcap_{\mathbf{M}}(X, \overline{X})$. The *branch-width* of $\mathbf{M}$, denoted by $\mathsf{bw}(\mathbf{M})$, is the $\lambda_{\mathbf{M}}$-width of $\mathbf{M}$, and its *path-width*, denoted by $\mathsf{pw}(\mathbf{M})$, is its linear $\lambda_{\mathbf{M}}$-width. It is easy to check that $\lambda_{\mathbf{M}}(X) \leq \mathsf{shadow}_{\mathbf{M}}(X)$ for any matroid $\mathbf{M}$ and $X \subseteq M$. As observed in [19], [11], [12], (path-width) branch-width and (linear) decomposition-width are essentially the same measure on matroids representable over a fixed finite field, *i.e.*, one is bounded by a function of the other.

Proposition II.2 ([19], [11], [12]). *For every $X \subseteq M$ of a matroid $\mathbf{M}$, $\lambda_{\mathbf{M}}(X) \leq \mathsf{shadow}_{\mathbf{M}}(X)$. Moreover, for each prime power q, there is a computable function f_q such that, if $\mathbf{M}$ is a $\mathsf{GF}(q)$-representable matroid, then $\mathsf{shadow}_{\mathbf{M}}(X) \leq f_q(\lambda_{\mathbf{M}}(X))$ for every subset $X \subseteq E(M)$.*

One can ask for which other matroids this is still the case. Funk, Mayhew and Newman proposed in [19] the following class of matroids. A class of matroids $\mathscr{C}$ is *strongly-pigeonhole* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every matroid $\mathbf{M} \in \mathscr{C}$ and every $X \subseteq M$, $\mathsf{shadow}_{\mathbf{M}}(X) \leq f(\lambda_{\mathbf{M}}(X))$.

[2] The root is never considered as a leaf. When S is a singleton $\{s\}$, the only rooted layout of S by convention is (T, δ) where T is the 2-node tree consisting in the root and a leaf $\delta(s)$.

[3] Matroids can be defined by several different set systems which are all equivalent. We will however stick here with the circuit-descriptions unless stated otherwise. We refer to [18] for more information on matroids.

By Proposition II.2, the class of matroids representable over a fixed finite field is strongly-pigeonhole. Other strongly-pigeonhole classes of matroids are the fundamental transversal matroids, 3-connected bicircular matroids, and 3-connected $\mathbb{H}$-gain graphic matroids with a finite group $\mathbb{H}$ [13].

For a matroid $\mathbf{M}$, and $S \subseteq M$, let $\mathbf{M}|_S$ be the set system with ground set S and set of hyperedges $\{C \in \mathcal{M} \mid C \subseteq S\}$, and let $\mathbf{M}/S$ be the set system with ground set $M \setminus S$ and whose set of hyperedges are the minimal elements of $\{C \mid \emptyset \neq C \subseteq M \setminus S$ and there is $T \subseteq S$ with $C \cup T \in \mathcal{M}\}$. It is well-known that both $\mathbf{M}|_S$ and $\mathbf{M}/S$ are matroids, *c.f.* [18, Chapter 3]. The following relate layouts of small $\lambda_{\mathbf{M}}$-width of a matroid $\mathbf{M}$ with layouts of small $\lambda_{\mathbf{N}}$-width of a *minor* $\mathbf{N} := (\mathbf{M}/A)|_B$ of $\mathbf{M}$ and will be used in order to prove that the tree produced by our transduction has small width.

Lemma II.3. *Let* $\mathbf{M}$ *be a matroid and let* U *be a subset of* M*. For every* $X \subseteq U$, $\lambda_{\mathbf{M}}(X) \leq \lambda_{\mathbf{M}}(U) + \sqcap_{\mathbf{M}}(X, U \setminus X)$.

Lemma II.4. *Let* $\mathbf{M}$ *be a matroid, and* A *and* B *be disjoint subsets of* M*. If* (T, δ) *is a rooted layout of* B *with* $\lambda_{(\mathbf{M}/A)|B}$*-width* c*, then this layout has* $\lambda_{\mathbf{M}|B}$*-width at most* $c + \sqcap_{\mathbf{M}}(B, A)$.

C. Monadic second-order logic

Define an *extended vocabulary* to be a finite set of relation and predicate names, each associated with an *arity* in $\mathbb{N}$. An *extended relational structure* $\mathbb{A}$ over an extended vocabulary Σ (Σ*-structure* for short) consists of a set A, called the *universe*, and, for each relation name R (resp. predicate name P) of Σ of arity r, a relation $\mathsf{R}^{\mathbb{A}} \subseteq A^r$ (resp. a predicate $\mathsf{P}^{\mathbb{A}} \subseteq (2^A)^r$). We use the following extended vocabularies for representing graphs, set systems, and layouts of set systems.

Graphs. A graph is represented by a relational $\{\mathsf{edg}\}$-structure $\mathbb{G}$ with the vertex set as universe and in which $\mathsf{edg}(x, y)$ holds if and only if xy is an edge. If in addition, each vertex is labeled with a label from a finite set Σ, called Σ-labeled graphs, then it can be represented by the $\{\mathsf{edg}, (\mathsf{V_a})_{a\in\Sigma}\}$-structure $\mathbb{G}$ where $x \in \mathsf{V_a}$ if x is labeled a.

Set systems. A set system is viewed as an extended relational $\{\mathsf{hyperedge}\}$-structure in which the universe of the structure is the ground set of the set system and the unary predicate $\mathsf{hyperedge}$ selects a set X whenever X is a hyperedge. For example, if a given set system is a matroid with circuits (independent sets, resp.) as hyperedges, the predicate $\mathsf{hyperedge}$ indicates whether X is a circuit (independent set, resp.).

Rooted layouts. A rooted layout of a set S is seen as a relational $\{\mathsf{prec}, \mathsf{map}\}$-structure in which the universe is the disjoint union of S and the set of nodes of the rooted tree. There are two binary relations, namely prec which is transitive and reflexive and which selects pairs (n, m) of tree nodes where n is a *precedent* of m (*i.e.*, a, possibly non-strict, ancestor of m), and map which is an injective function selecting pairs (x, n) whenever $x \in S$ is mapped to the leaf n of the rooted tree. Tree nodes can be recovered as those elements that are their own precedent (by reflexivity of prec) and elements of S can be recovered as the rest of the universe.

Rooted layouts of (extended) relational structures. A rooted layout of an (extended) relational structure will be represented by the union of the (extended) relational structure and the relational structure representing the rooted layout of its ground set. For instance, the rooted layout of a set system is a $\{\mathsf{prec}, \mathsf{map}, \mathsf{hyperedge}\}$-structure ($\mathbf{S}\cup\mathsf{T}$), where S is the subset of the universe of T which are not tree nodes and which is mapped to nodes according to map.

To describe properties of relational structures, we use *monadic second order logic* (*MSO* logic). We refer to [10] for a complete presentation of MSO logic. This logic has two types of variables, namely FO *variables* (denoted with lower-case letters) and *set variables* (denoted with upper-case letters), that can be quantified both universally and existentially. A variable is *free* in a formula if it is not bound by a quantifier. We write $\varphi(x_1, \ldots, x_p, X_1, \ldots, X_q)$ to say that $x_1, \ldots, x_p$ and $X_1, \ldots, X_q$ are among the free variables of φ. An *MSO sentence* is an MSO formula without free variables. If a sentence ϕ is true in a structure $\mathbb{A}$, we say that $\mathbb{A}$ *models* ϕ, written as $\mathbb{A} \models \phi$. For an extended vocabulary Σ, let us denote by $\mathbf{S}^\Sigma$ the set of relational structures over the vocabulary Σ. A *class of relational structures over* Σ is a subset $\mathscr{C}$ of $\mathbf{S}^\Sigma$ which is closed under isomorphism. It is *MSO-definable* if there is an MSO sentence $\varphi_{\mathscr{C}}$ such that $\mathbb{A} \in \mathscr{C}$ if and only if $\mathbb{A} \models \varphi_{\mathscr{C}}$.

D. Monadic second-order transductions

We may also use MSO logic to describe transformations of relational structures. Such transformations are described in terms of *MSO-transduction*, that we now briefly introduce. We refer to [10] for a detailed introduction, and we use the equivalent presentation given in [2] in which transductions are divided into "atomic" operations – the class of MSO-transductions being indeed closed under composition [10, Theorem 1.39]. Given two (extended) vocabularies Σ and Γ, define a Σ*-to-*Γ *MSO-transduction* as a set of pairs in $\mathbf{S}^\Sigma \times \mathbf{S}^\Gamma$, which can be defined using the MSO logic, as a composition of the three following *atomic transductions*:

Copying: For each integer k, define the k-copying of a Σ-structure $\mathbb{A}$ to be k disjoint copies of $\mathbb{A}$, with the following fresh predicates added to the vocabulary: $\mathsf{copy}(x, y)$, and, for each $i \in [k]$, $\mathsf{layer}_i(x)$. The binary predicate copy checks whether two elements are copies of the same element of the original structure, whereas the unary predicate layer_i checks whether an element belongs to the i-th copy (or i-th layer).

Coloring: For each integer $p \geq 1$, define the p-coloring of a Σ-structure $\mathbb{A}$ to be any structure obtained from $\mathbb{A}$ by adding p unary relations $\mathsf{X}_1, \ldots, \mathsf{X}_p$ to the vocabulary and interpreting them as any subset of the universe.

Interpreting: The syntax of an MSO-interpretation consists in an input vocabulary Σ, an output vocabulary Γ, and the following family of MSO formulas over Σ: φ_{dom}, $\varphi_{\mathsf{univ}}(x)$, and, for each relation name R (resp. predicate name P) of arity r, a formula $\varphi_R(x_1, \ldots, x_r)$ (resp. $\varphi_{\mathsf{P}}(X_1, \ldots, X_r)$). The formula φ_{dom} is a sentence, which filters the inputs of the transduction. More precisely, a Σ-structure $\mathbb{A}$ will admit an output by the interpretation if and only if $\mathbb{A} \models \varphi_{\mathsf{dom}}$. When this is the case, the output is defined as the Γ-structure whose universe is the restriction of the input universe to those elements x that satisfy $\varphi_{\mathsf{univ}}(x)$, and in which each relation name R (resp. predicate P) is interpreted as those tuples $(x_1, \ldots, x_r)$ (resp. $(X_1, \ldots, X_r)$) that satisfy the corresponding formula $\varphi_R(x_1, \ldots, x_r)$ (resp. $\varphi_{\mathsf{P}}(X_1, \ldots, X_r)$).

Interpreting is a partial function, copying is a total function, and coloring is not a function (but has at least one output for each input). A corollary of [10, Theorem 7.14] says that for every Σ-to-Γ MSO-transduction τ, there are MSO-formulas on Σ describing $\tau(\mathbb{M})$ from $\mathbb{M}$. We say that an MSO-transduction is *effectively given* if these formulas are computable. The *quantifier height* of the transduction is the maximum quantifier height of these formulas. There are two other measures on MSO-transductions: the number of copies, and the number of parameters (*i.e.*, the number of used colors). We speak of *k-copying p-parameter MSO-transductions of quantifier height h*, and we refer to [10] for precise definitions.

One of the most important features of MSO-transductions is the *Backwards Translation Theorem* [10, Theorem 1.40]. It says that the pre-image of an MSO-definable class by an MSO-transduction is itself MSO-definable. Another important property of MSO-transductions is the *Parallel Application Lemma* [5] that shows how an MSO-transduction may act in parallel on an unbounded number of disjoint structures. Let Σ be a vocabulary and $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be disjoint Σ-structures. Define the *disjoint union* of structures $\mathbb{A}_1, \ldots, \mathbb{A}_n$, denoted by $\bigsqcup_{0<i\leq n} \mathbb{A}_i$, as the following structure over the vocabulary $\Sigma \cup \{\sim\}$ where $\sim$ is a new binary relation name:

- the universe of $\bigsqcup_{0<i\leq n} \mathbb{A}_i$ is the union of the universes of the $\mathbb{A}_i$s (which are required to be disjoint);
- for each relation or predicate name from Σ, its interpretation in $\bigsqcup_{0<i\leq n} \mathbb{A}_i$ is the union of its interpretations in each of the $\mathbb{A}_i$s;
- the interpretation of $\sim$ in $\bigsqcup_{0<i\leq n} \mathbb{A}_i$ is the set of pairs of elements that originate from the same $\mathbb{A}_i$.

Lemma II.5 (Parallel Application Lemma [5])**.** *Let τ be a k-copying p-parameters Σ-to-Γ MSO-transduction of quantifier-height q. Then there is a k-copying p-parameters $(\Sigma \cup \{\sim\})$-to-$(\Gamma \cup \{\sim\})$ MSO-transduction $\hat{\tau}$, of quantifier-height* $\max\{q, p+1\}$*, such that, for every sequence $\mathbf{I}_1, \ldots, \mathbf{I}_n$ of Σ-structures and every sequence $\mathbf{O}_1, \ldots, \mathbf{O}_n$ of Γ-structures, we have $(\bigsqcup_{0<i\leq n} \mathbf{I}_i, \bigsqcup_{0<i\leq n} \mathbf{O}_i) \in \hat{\tau}$ if and only if there exists a permutation π of $[n]$ such that $(\mathbf{I}_i, \mathbf{O}_{\pi(i)}) \in \tau$ for all $i \in [n]$.*

E. Recognisability

We follow the terminology of [10]. A *functional signature* is a set of function symbols such that each function symbol f is given with a nonnegative integer, the *arity of f*, denoted by *$ar(f)$*. Function symbols of arity 0 are called *constants*. The set of *terms of F*, denoted by *$T(F)$*, is the set defined inductively as:

base: each constant symbol is a term in $T(F)$;

rule: if f is a function symbol of arity k and $t_1, \ldots, t_k$ are terms in $T(F)$, then $f(t_1, \ldots, t_k)$ is a term in $T(F)$.

For a functional signature F, an *F-algebra* $\mathbb{M}$ is a set M equipped with total functions $f_{\mathbb{M}} : M^{\text{ar}(f)} \to M$, for all $f \in F$. Each term $t \in T(F)$ is associated with an element in M, denoted by $\mathbf{val}_{\mathbb{M}}(t)$, and defined inductively as:

base: if t is a constant symbol c, then $\mathbf{val}_{\mathbb{M}}(t)$ is $c_{\mathbb{M}}$;

rule: if t is $f(t_1, \ldots, t_k)$, then $\mathbf{val}_{\mathbb{M}}(t)$ is $f_{\mathbb{M}}(\mathbf{val}_{\mathbb{M}}(t_1), \ldots, \mathbf{val}_{\mathbb{M}}(t_k))$.

If $\mathbb{M}$ is an F-algebra and each element of M is equal to $\mathbf{val}_{\mathbb{M}}(t)$ for some term t in $T(F)$, then we say that $\mathbb{M}$ is *generated by F*.

We denote by $\mathbb{T}(F)$ the F-algebra where for each $f \in F$, the function $f_{\mathbb{T}(F)}$ is defined as

$$f_{\mathbb{T}(F)}(t_1, \ldots, t_{\text{ar(f)}}) = f(t_1, \ldots, t_{\text{ar(f)}}).$$

For a functional signature F and two F-algebras $\mathbb{M}$ and $\mathbb{A}$, a function $h : M \to A$ is an *F-algebra homomorphism* if for every $f \in F$ and $m_1, \ldots, m_{\text{ar}(f)}$ in M, we have $h(f_{\mathbb{M}}(m_1, \ldots, m_{\text{ar}(f)})) = f_{\mathbb{A}}(h(m_1), \ldots, h(m_{\text{ar}(f)}))$. If $\mathbb{M}$ is an F-algebra, a subset L of M is said *recognisable* in $\mathbb{M}$ if $L = h^{-1}(C)$, where $h : \mathbb{M} \to \mathbb{A}$ is an F-algebra homomorphism,[4] A is finite and $C \subseteq A$.

The following result from [10] states that recognisability of terms is the same as recognisability of subsets of algebras.

Proposition II.6 ([10, Proposition 3.69])**.** *Let $\mathbb{M}$ be an F-algebra generated by a functional signature F and let L be a subset of M. Then, L is recognisable in $\mathbb{M}$ if and only if $\mathbf{val}_{\mathbb{M}}^{-1}(L)$ is recognisable in $\mathbb{T}(F)$.*

Let's now describe the algebra that generates set systems of decomposition-width k, that is implicit in [19, Lemma 5.2] and extends the algebra defined in [11], [12] for matroids. Let k be a positive integer. A *k-ported set system* is a pair $(\mathbf{S}, \rho)$ with $\mathbf{S}$ a set system and $\rho : 2^S \to [k] \cup \{\mathbf{indep}, \mathbf{dep}\}$ a function such that $\rho(X) = \mathbf{indep}$ only if $X \in \mathcal{S}$. Let *PS_k* be a functional signature consisting of $(k+2)^{(k+2)^2}$ binary function symbols and 16 constant symbols.

For two k-ported set systems $(\mathbf{S}_1, \rho_1)$, $(\mathbf{S}_2, \rho_2)$ and a function $f : ([k] \cup \{\mathbf{indep}, \mathbf{dep}\})^2 \to [k] \cup \{\mathbf{indep}, \mathbf{dep}\}$, let $(\mathbf{S}_1, \rho_1) \oplus_f (\mathbf{S}_2, \rho_2)$ be the k-ported set system $(\mathbf{S}, \rho)$ where S is the disjoint union of S_1 and S_2 and

$$\mathcal{S} = \{X_1 \cup X_2 \mid X_1 \subseteq S_1,\ X_2 \subseteq S_2, f(\rho_1(X_1), \rho_2(X_2)) = \mathbf{indep}\},$$
$$\rho(X) = f(\rho_1(X \cap S_1), \rho_2(X \cap S_2)) \qquad \text{for all } X \subseteq S.$$

For each $f : \{0,1\} \to \{1, 2, \mathbf{indep}, \mathbf{dep}\}$, let $\mathbf{c}_f$ be the k-ported set system $(\mathbf{S}, \rho)$ with S a singleton, $f^{-1}(\mathbf{indep})$ as hyperedges, and

$$\rho(X) = \begin{cases} f(0) & \text{if } X = \emptyset, \\ f(1) & \text{if } X = S. \end{cases}$$

[4]The definition given in [10, Definition 3.55] is more general and allows algebras with infinitely many sorts, but we restrict ourselves to the case with one sort which is enough for our purposes.

Let's denote by $\mathbb{PS}_k$ the PS_k-algebra with binary function symbols bijectively mapped to the set of binary operations $\{\oplus_f \mid f : ([k] \cup \{\mathbf{indep}, \mathbf{dep}\})^2 \to [k] \cup \{\mathbf{indep}, \mathbf{dep}\}\}$ and constant symbols bijectively mapped to the set $\{\mathbf{c}_f \mid f : \{0,1\} \to \{1, 2, \mathbf{indep}, \mathbf{dep}\}\}$. It is clear that each term in $T(PS_k)$ evaluates into a k-ported set system.

Proposition II.7 ([19, Lemma 5.2])**.** *Let k be a fixed integer. There exists a deterministic MSO-transduction $\mathbf{val}_{\mathbb{PS}_k}$ such that if (T, δ) is a rooted binary layout of $\mathsf{shadow}_\mathbf{S}$-width k of a set system $\mathbf{S}$, then one can label every node of T to obtain a term t in $T(PS_k)$ so that $\mathbf{val}_{\mathbb{PS}_k}(t)$ is isomorphic to $\mathbf{S}$.*

It is not hard to check that if a set system $\mathbf{S}$ is isomorphic to $\mathbf{val}_{\mathbb{PS}_k}(t)$ for some term t in $T(PS_k)$, then $\mathsf{dw}(\mathbf{S}) \le k$. One can therefore use the PS_k-algebra $\mathbb{PS}_k$ to define the notion of recognisability for set systems. A class $\mathscr{C}$ of set systems is said to be *k-recognisable* if the set $\{\mathbf{S} \mid \mathbf{S} \in \mathscr{C}$ and $\mathsf{dw}(\mathbf{S}) \le k\}$ is recognisable in $\mathbb{PS}_k$, and it is *recognisable* if it is k-recognisable for each integer $k \ge 0$. The following can be obtained by combining results from [10] and Proposition II.7.

Theorem II.8. *If a class of set systems $\mathscr{C}$ is CMSO-definable, then it is recognisable.*

III. Main result

A natural question with regard to Theorem II.8 is whether its converse is true, generalizing [20, Conjecture 1] which concerns only graphs, to all set systems including matroids. Our main result, stated below, is a partial answer, which implies the results proved in [5] concerning graphs of bounded linear clique-width, and those from [2] concerning graphs of bounded path-width. We now state our main result.

Theorem III.1. *Let $\mathbb{F}$ be a finite field and let $\mathscr{C}$ be a class of $\mathbb{F}$-representable matroids. If $\mathscr{C}$ is recognisable, then the subclass of $\mathscr{C}$ consisting of all matroids of path-width at most k is CMSO-definable, for each k.*

The key ingredient for proving Theorem III.1 is a CMSO-transduction which, given a $\mathbb{F}$-representable matroid $\mathbf{M}$ of path-width at most k, nondeterministically produces a rooted layout of $\mathbf{M}$ of decomposition-width at most $f(k)$, for some function f. We actually prove a stronger result, where $\mathbf{M}$ does not need to be $\mathbb{F}$-representable, but only to come from a strongly-pigeonhole class of matroids.

Theorem III.2. *Let k be a positive integer and let $g : \mathbb{N} \to \mathbb{N}$ be a function. Then, there exists a constant c (depending only on k and $g(h(k))$ for some computable function h), and an MSO-transduction τ from matroids to rooted layouts of matroids such that, on every input matroid $\mathbf{M}$, the following holds:*

1) *if $\mathbf{M}$ is a matroid of path-width at most k such that $\mathsf{shadow}_\mathbf{M}(X) \le g(\lambda_\mathbf{M}(X))$ for every $X \subseteq M$, then τ outputs at least one rooted layout of $\mathbf{M}$ of $\lambda_\mathbf{M}$-width at most c, and*
2) *every output of τ is a rooted layout of $\mathbf{M}$ of $\lambda_\mathbf{M}$-width at most c.*

The rest of this section is devoted to the proof of Theorem III.1, assuming Theorem III.2. The proof of Theorem III.2 is the purpose of the next section.

Using Theorem III.2, we can prove Theorem III.1 which is a partial converse to Theorem II.8. However, transducing layouts is not enough to prove the definability. While this task is easy when layouts are binary [5], this is less immediate in our case because the layouts output by Theorem III.2 are unranked. To overcome this difficulty, we first transfer the recognisability of input matroids to the recognisability of transduced layouts. To this end we first prove that we can label rooted layouts with a finite alphabet so that, for any ordering of the nodes of the tree, there is an MSO-transduction outputting a labeled binary tree evaluating into $\mathsf{GF}(q)$-representable matroids. This labeling is done by relating $\mathsf{GF}(q)$-representable matroids of small branch-width with graphs of small clique-width, and such a relation between $\mathsf{GF}(q)$-representable matroids and graphs was already known for binary matroids, see for instance [8], relation used for instance to obtain as a corollary of Theorem III.1 (see Corollary III.6) the equivalence between recognisability and CMSO-definability for graph classes of bounded *linear clique-width*, result proved in [5].

Proposition III.3. *Let q be a prime power and let k be a fixed positive integer. Then, there are a finite alphabet Σ_k of size at most $g(q,k)$, for some computable function g, an MSO-transduction τ_1 and two deterministic MSO-transductions τ_2 and β such that, for every $\mathsf{GF}(q)$-representable matroid $\mathbf{M}$ and every rooted layout T of $\mathbf{M}$ of $\lambda_\mathbf{M}$-width k,*

1) *every output $t \in \tau_1(\mathbf{M} \cup \mathsf{T})$ is a Σ_k-labeled tree such that $\tau_2(t) = \mathbf{M}$;*
2) *$\beta(t, \le)$, for every $t \in \tau_1(\mathbf{M} \cup \mathsf{T})$, is a term t' in $T(PS_{h(g(q,k))})$ such that $\mathbf{val}_{\mathbb{PS}_{h(g(q,k))}}(t') = \mathbf{M}$, where $\le$ is any linear ordering of the nodes of T and h is a computable function.*

Secondly, we use an old result by Courcelle linking recognisability of graphs with *order-invariant MSO-definability*. An *ordered graph* is a graph with a linear ordering[6] We call a class $\mathcal{L}$ of ordered graphs *order-invariant* if, for any graph G and any two linear orderings $\le_1$ and $\le_2$, it holds that $(G, \le_1) \in \mathcal{L}$ if and only if $(G, \le_2) \in \mathcal{L}$. If $\le$ is a binary relation symbol, let's denote by $\mathsf{MSO}_1(\le)$ the set of monadic second-order formulas over the relational signature $\{\mathbf{edg}, \le\}$. A property $\mathcal{L}$ on graphs is said *$\mathsf{MSO}_1(\le)$-expressible* if the class of ordered graphs $\{(G, \le) \mid G \in \mathcal{L},\ \le$ a linear ordering of $G\}$ is an order-invariant MSO_1-definable property.

Theorem III.4 ([21, Theorem 4.1])**.** *Any $\mathsf{MSO}_1(\le)$-expressible graph property is recognisable.*

[5]When the rooted layouts are binary, there is an MSO-transduction that guesses the operations from PS_c in order to output a term in $T(PS_c)$ evaluating into matroid inputs, where c is the constant from Theorem III.2.

[6]The definition extends to any relational structure, but we stick to graphs as we only need the result on graphs.

Now, a combination of Proposition III.3 and of Theorem III.4 will allow to transfer the recognisability of input matroids to the labeled trees output by Proposition III.3. To obtain the CMSO-definability, we use the following result stating that on labeled trees recognisability is the same as CMSO-definability.

Proposition III.5 ([1, Sections 4,5])**.** *Let Σ be a finite alphabet. A family of Σ-labeled trees is recognisable if and only if it is CMSO-definable.*

We are now ready to prove Theorem III.1.

Proof of Theorem III.1: Let q and k be fixed positive integers. Let $\mathscr{M}$ be a recognisable class of $\mathsf{GF}(q)$-representable matroids of path-width at most k. By Theorem III.2, there is an MSO-transduction τ that, given a matroid $\mathbf{M}$ in $\mathscr{M}$, outputs a rooted layout of $\lambda_{\mathbf{M}}$-width at most $g(k)$, for some computable function g. By Proposition III.3 there is an MSO-transduction τ' such that $\tau' \circ \tau$ outputs a set of Σ_k-labeled rooted trees, for some finite alphabet Σ_k. Let $\mathcal{L}_k$ be the set of Σ_k-labeled trees output by the MSO-transduction $\tau' \circ \tau$ on inputs matroids from $\mathscr{M}$. Again by Proposition III.3, there is an MSO-transduction β that takes as input $\mathbb{T}$ in $\mathcal{L}_k(\leq)$ and outputs a term t in $T(PS_{g(q,k)})$, for some computable function g and such that $\mathbf{val}_{\mathbb{PS}_{g(q,k)}}(t) = \mathbf{M}$. Because $\mathscr{M}$ is recognisable, we can conclude that $\beta(\mathcal{L}_k(\leq))$ is also recognisable by Proposition II.6, Thus by Proposition III.5, $\beta(\mathcal{L}_k(\leq))$ is CMSO-definable. Since, for $\mathbb{T} \in \tau'(\tau(\mathbf{M}))$ and any two linear orderings $\leq_1$ and $\leq_2$ of $\mathbb{T}$, $\beta((\mathbb{T}, \leq_1)) \in \beta(\mathcal{L}_k(\leq))$ if and only if $\beta((\mathbb{T}, \leq_2)) \in \beta(\mathcal{L}_k(\leq))$, we can conclude that $\mathcal{L}_k$ is $\mathsf{MSO}_1(\leq)$-expressible.[7] Hence, by Theorem III.4 $\mathcal{L}_k$ is recognisable, and then by Proposition III.5 $\mathcal{L}_k$ is CMSO-definable. By the Backwards Translation Theorem [10, Theorem 1.40], we can conclude that $\mathscr{M}$ is CMSO-definable. ■

As a corollary we obtain the result from [5] concerning graph classes of bounded linear clique-width and then the first result in [2] concerning graph classes of bounded path-width.

Corollary III.6. *Let $\mathscr{C}$ be a class of graphs. Then, on subclasses of $\mathscr{C}$ of bounded linear clique-width recognisability equals CMSO-definability.*

Proof: From any graph $\mathbf{G}$, one can associate the binary matroid $\mathbf{M}(G)$ represented by the matrix $(I_{V(\mathbf{G})} \;\; A)$ over the binary field $\mathbb{F}_2$ where A is the adjacency matrix of $\mathbf{G}$ and $I_{V(\mathbf{G})}$ is the identity matrix (rows are indexed by $V(\mathbf{G})$ and first n columns by $V(\mathbf{G})$ and last n columns by a copy of $V(\mathbf{G})$). One easily checks that the matroid $\mathbf{M}(\mathbf{G})$ can be computed from $\mathbf{G}$ by an MSO-transduction. It is known that the path-width of $\mathbf{M}$ is 2 times the linear clique-width of $\mathbf{G}$ and one can compute $\mathbf{G}$ by an MSO-transduction from $\mathbf{M}$ (see for instance [8]). Now, one checks that if $\mathscr{C}$ is recognisable, then $\{\mathbf{M}(\mathbf{G}) \mid \mathbf{G} \in \mathscr{C}\}$ is also recognisable. By Theorem III.1, the subset of $\{\mathbf{M}(\mathbf{G}) \mid \mathbf{G} \in \mathscr{C}\}$ of path-width $2k$ is CMSO-definable, hence the subset of $\mathscr{C}$ of linear clique-width k is CMSO-definable. ■

[7] It is worth mentioning that counting predicates can be expressed using the ordering $\leq$, that is why we write $\mathsf{MSO}_1(\leq)$ instead of $\mathsf{CMSO}_1(\leq)$.

Remark. While many of the constructions seem to work for set systems of small linear decomposition-width, there are three main technical issues on extending Theorem III.1 to them. Firstly, a notion of circuit on set systems that behave well with respect to (linear) decomposition-width is unclear. Secondly, while we can circumvent it on the special case of *independent set systems* – matroids are a special case – by defining circuits as *minimal transversals* of hyperedges, we cannot even prove lemmas similar to Lemmas II.3 and II.4 with the shadow function on independent set systems. This is the reason why we restrict ourselves to strongly-pigeonhole matroids in Theorem III.2. Thirdly, even if we circumvent these two obstacles, we need to prove that any layout (T, δ) of $\mathsf{shadow}_{\mathbf{S}}$-width k of a set system $\mathbf{S}$ can be labeled so that the set $\mathcal{S}$ is CMSO-definable from the labeled one. Unfortunately we can prove such a labeling only for $\mathsf{GF}(q)$-representable matroids, with q a prime power (see Proposition III.3), reason why Theorem III.1 is only stated for them.

IV. Proof of Theorem III.2 : Definability for bounded path-width

We are now ready to propose an MSO-transduction for describing a rooted layout of small λ-width for strongly-pigeonhole matroids. We recall that there is no hope to describe a linear layout of small λ-width, otherwise we might be able to describe in MSO a linear ordering on any set. As in [5], [2], our proof relies on Simon's Factorisation Forest Theorem which will be used to color the universe with a bounded number of colors and use the Parallel Application Lemma in order to construct the transduction. Let's first recall Simon's Factorisation Forest Theorem.

Remind that a *semi-group* is a set S equipped with an associative binary operation. Notice also that A^* is the set of finite words over the alphabet A, while A^+ is the set of non-empty finite words over A, and each equipped with concatenation $\cdot$ is a semi-group. An *idempotent element* in a semi-group $(S, \circ)$ is an element e such that $e \circ e = e$. For two semi-groups $(S_1, \circ_1)$ and $(S_2, \circ_2)$, a *semi-group homomorphism* is a function $h : S_1 \to S_2$ such that $h(x \circ_1 y) = h(x) \circ_2 h(y)$.

Let $(S, \circ)$ be a semi-group and A an alphabet. For a semi-group homomorphism $h : A^+ \to S$, an *h-factorisation* of a word $w \in A^*$ is a sequence $(w_1, \ldots, w_n)$ such that

1) $w = w_1 \cdot w_2 \cdot \cdots \cdot w_n$,
2) $|w_i| < |w|$ for all $i \in [n]$, and
3) $h(w_1) = h(w_2) = \cdots = h(w_n)$ is idempotent if $n \geq 3$.

The *h-rank* of a word $w \in A^*$ is defined inductively as follows : single letters have h-rank 1, and for every $w \in A^*$ of length at least 2, its h-rank is

$$1 + \min_{(w_1,\ldots,w_n) \text{ is an } h\text{-factorisation of } w} \left(\max_{1 \leq i \leq n} \{h\text{-rank of } w_i\} \right).$$

Imre Simon proved in [22] that the h-rank of any word is upper-bounded by a function on the size of the target semi-group, which we refer below with the improvement given in [23].

Theorem IV.1 (Simon's Factorisation Forest Theorem [23])**.** *Let S be a finite semi-group and let $h : A^* \to S$ be a semi-group homomorphism. Then, every word $w \in A^+$ has h-rank at most $3 \cdot |S|$.*

A. Bi-ported set systems

We will construct the transduction using an induction based on Simon's Factorisation Theorem. For doing so, we need to associate with each strongly-pigeonhole matroid a word in a word language and a finite semi-group recognising the associated word language. We propose here an algebra, based on the functional signature PS_k, $k \geq 1$, and suited for linear decomposition-width. We first prove that any set system of linear decomposition-width k can be associated with a word using the functions of the algebra. Then, we show that in case of matroids, one can furthermore require some more properties on the words. One can check that our algebra will not only resume boundaried $\mathsf{GF}(q)$-representable matroids of bounded branch-width [9], [11], [12], but also bi-interfaced graphs considered in [2].

Let L and R be prescribed sets of size k with distinguished elements $\mathsf{l}^\emptyset \in \mathrm{L}$ and $\mathsf{r}^\emptyset \in \mathrm{R}$. We call L and R the *left labels* and *right labels*, respectively. If $\mathsf{shadow}_{\mathbf{S}}(U)$ and $\mathsf{shadow}_{\mathbf{S}}(\overline{U})$ are less than or equal to k, then we may use subsets of L and R to label the equivalence classes coming from the left and right respectively. When this is done, we reserve the distinguished elements $\mathsf{l}^\emptyset$ and $\mathsf{r}^\emptyset$ for the equivalence class that contains the empty set. This motivates the following structure.

Definition IV.2. *A* bi-ported set system *is a tuple $\mathbb{S} = (\mathbf{I}^{\mathrm{L}}, S, \mathbf{I}^{\mathrm{R}}; \mathcal{S})$ where*

1) *S is a finite set called the* internal elements*,*
2) *$\mathbf{I}^{\mathrm{L}}$ and $\mathbf{I}^{\mathrm{R}}$ are subsets of $\mathrm{L}\times\mathrm{R}$, called the* left interactions *and* right interactions*, respectively, and*
3) *$\mathcal{S}$ is a subset of $\mathbf{I}^{\mathrm{L}} \times 2^S \times \mathbf{I}^{\mathrm{R}}$ called the* hyperedges*, such that: a*
   a) *For each $\mathbf{i}^{\mathrm{L}} \in \mathbf{I}^{\mathrm{L}}$, there exist $Z \subseteq S$ and $\mathbf{i}^{\mathrm{R}} \in \mathbf{I}^{\mathrm{R}}$ such that $(\mathbf{i}^{\mathrm{L}}, Z, \mathbf{i}^{\mathrm{R}}) \in \mathcal{S}$. Similarly, for each $\mathbf{i}^{\mathrm{R}} \in \mathbf{I}^{\mathrm{R}}$, there exist $Z \subseteq S$ and $\mathbf{i}^{\mathrm{L}} \in \mathbf{I}^{\mathrm{L}}$ such that $(\mathbf{i}^{\mathrm{L}}, Z, \mathbf{i}^{\mathrm{R}}) \in \mathcal{S}$.*
   b) *There is a* rightwards relabelling *function $\rho^{\mathrm{R}} : \mathrm{L}\times 2^S \to \mathrm{L}$ and a* leftwards relabelling *function $\rho^{\mathrm{L}} : 2^S \times \mathrm{R} \to \mathrm{R}$ such that $((\mathsf{l}^{\mathrm{L}}, \mathsf{r}^{\mathrm{L}}), Z, (\mathsf{l}^{\mathrm{R}}, \mathsf{r}^{\mathrm{R}})) \in \mathbf{I}^{\mathrm{L}} \times 2^S \times \mathbf{I}^{\mathrm{R}}$ is a hyperedge only if $\rho^{\mathrm{R}}(\mathsf{l}^{\mathrm{L}}, Z) = \mathsf{l}^{\mathrm{R}}$ and $\mathsf{r}^{\mathrm{L}} = \rho^{\mathrm{L}}(Z, \mathsf{r}^{\mathrm{R}})$. Furthermore $\rho^{\mathrm{R}}(\mathsf{l}^\emptyset, \emptyset) = \mathsf{l}^\emptyset$ and $\mathsf{r}^\emptyset = \rho^{\mathrm{L}}(\emptyset, \mathsf{r}^\emptyset)$.*

   Any hyperedge $((\mathsf{l}^{\mathrm{L}}, \mathsf{r}^{\mathrm{L}}), \emptyset, (\mathsf{l}^{\mathrm{R}}, \mathsf{r}^{\mathrm{R}}))$ such that $\mathsf{l}^{\mathrm{L}} \neq \mathsf{l}^\emptyset$ and $\mathsf{r}^{\mathrm{R}} \neq \mathsf{r}^\emptyset$ is called a trivial hyperedge*.*

Notice that for each $X \subseteq S$, $\mathsf{l}^{\mathrm{L}} \in \mathrm{L}$ and $\mathsf{r}^{\mathrm{R}} \in \mathrm{R}$, there are unique $\mathsf{r}^{\mathrm{L}} \in \mathrm{R}$ and $\mathsf{l}^{\mathrm{R}} \in \mathrm{L}$ such that $((\mathsf{l}^{\mathrm{L}}, \mathsf{r}^{\mathrm{L}}), X, (\mathsf{l}^{\mathrm{R}}, \mathsf{r}^{\mathrm{R}})) \in \mathcal{S}$. If $\mathbf{S}$ is a set system and (A, B, C) is a tri-partition of S, we would like bi-ported set systems on A, B, and C, which describe how to combine subsets in A, B, and C in order to obtain a set in $\mathcal{S}$. To do so, we introduce a concatenation operation on bi-ported set systems.

For two bi-ported set systems $\mathbb{S}_1 = (\mathbf{I}^{\mathrm{L}}_1, S_1, \mathbf{I}^{\mathrm{R}}_1; \mathcal{S}_1)$ and $\mathbb{S}_2 = (\mathbf{I}^{\mathrm{L}}_2, S_2, \mathbf{I}^{\mathrm{R}}_2; \mathcal{S}_2)$ with $\mathbf{I}^{\mathrm{R}}_1 = \mathbf{I}^{\mathrm{L}}_2$, we define the *concatenation* $\mathbb{S}_1 \odot \mathbb{S}_2$ as the bi-ported set system $\mathbb{S} = (\mathbf{I}^{\mathrm{L}}_1, S_1 \sqcup S_2, \mathbf{I}^{\mathrm{R}}_2; \mathcal{S})$ where the elements of $\mathcal{S}$ are of the form $(\mathbf{i}_1, Z_1 \sqcup Z_2, \mathbf{i}_2)$ for some $(\mathbf{i}_1, Z_1, \mathbf{i}_3) \in \mathcal{S}_1$ and $(\mathbf{i}_3, Z_2, \mathbf{i}_2) \in \mathcal{S}_2$. If $\mathbf{I}^{\mathrm{R}}_1 \neq \mathbf{I}^{\mathrm{L}}_2$, we instead define $\mathbb{S}_1 \odot \mathbb{S}_2$ as a new object $\mathbf{0}$. We think of $\mathbf{0}$ as denoting "syntactical error". We also define $\mathbb{S}_1 \odot \mathbf{0} = \mathbf{0} \odot \mathbb{S}_2 = \mathbf{0}$. The following tells that the concatenation operation on bi-ported set systems is well-defined.

Lemma IV.3. *If $\mathbb{S}_1 = (\mathbf{I}^{\mathrm{L}}_1, S_1, \mathbf{I}^{\mathrm{R}}_1; \mathcal{S}_1)$ and $\mathbb{S}_2 = (\mathbf{I}^{\mathrm{L}}_2, S_2, \mathbf{I}^{\mathrm{R}}_2; \mathcal{S}_2)$ are bi-ported set systems with $\mathbf{I}^{\mathrm{R}}_1 = \mathbf{I}^{\mathrm{L}}_2$, then their concatenation $\mathbb{S} = \mathbb{S}_1 \odot \mathbb{S}_2$ is also a bi-ported set system and is unique.*

For a positive integer $k \geq 2$, we let $\mathbf{BSS}_k$ be the set of bi-ported set systems with $\mathrm{L} = \mathrm{R} = [k]$, and with exactly one internal element. If $\mathbb{S}_1\mathbb{S}_2 \ldots \mathbb{S}_p$ is a word on the alphabet $\mathbf{BSS}_k$, its value is the bi-ported set system $\mathbb{S}_1 \odot \mathbb{S}_2 \odot \cdots \odot \mathbb{S}_p$.

Let $\mathbb{S}$ be a bi-ported set system. We say $(\mathsf{l}^{\mathrm{L}}, \mathsf{r}^{\mathrm{L}}) \in \mathbf{I}^{\mathrm{L}}$ is *internally realised by $Z \subseteq S$* when $\mathsf{r}^{\mathrm{L}} = \rho^{\mathrm{L}}(Z, \mathsf{r}^\emptyset)$. Similarly, we say $(\mathsf{l}^{\mathrm{R}}, \mathsf{r}^{\mathrm{R}}) \in \mathbf{I}^{\mathrm{R}}$ is *internally realised by $Z \subseteq S$* when $\mathsf{l}^{\mathrm{R}} = \rho^{\mathrm{R}}(\mathsf{l}^\emptyset, Z)$. A subset X of S is called an *internal hyperedge* in $\mathbb{S}$ if there are $\mathsf{r}^{\mathrm{L}} \in \mathrm{R}$ and $\mathsf{l}^{\mathrm{R}} \in \mathrm{L}$ that are both internally realised by X, *i.e.*, if $((\mathsf{l}^\emptyset, \mathsf{r}^{\mathrm{L}}), X, (\mathsf{l}^{\mathrm{R}}, \mathsf{r}^\emptyset))$ is in $\mathcal{S}$. By abuse of language we talk about the *decomposition-width of a bi-ported set system* $\mathbb{S}$ as the decomposition-width of the set system with ground set S and set of internal hyperedges of $\mathbb{S}$ as set of hyperedges. If a set system $\mathbf{S}$ has linear decomposition-width at most k, then it is the value of a word on the alphabet $\mathbf{BSS}_{2^k}$.

Proposition IV.4. *Let $\mathbf{S} = (S, \mathcal{S})$ be a set system.*

1) *If $\mathbf{S}$ is the value of a word on the alphabet $\mathbf{BSS}_k$, then it has linear decomposition-width at most k.*
2) *If $\mathbf{S}$ has linear decomposition-width at most k, then it is the value of a word on the alphabet $\mathbf{BSS}_{2^k}$.*

While Proposition IV.4 will allow associating with set systems of linear decomposition-width k a semi-group recognising them (see next Section), we unfortunately need some properties that allow us to control how internal hyperedges are constructed during concatenation. These properties are subsumed by the following equivalence relation, that can be defined on any set system satisfying the two first axioms of circuit-set axiom of matroids. We however define it only for matroids avoiding the definition of circuits for set systems.

For a matroid $\mathbf{M}$ and subset U of M, we let $\simeq^U_{\mathbf{M}}$ be the binary relation on $2^U \times 2^U$ where, for each two subsets X and Y of U, $X \simeq^U_{\mathbf{M}} Y$ if $X \equiv^U_{\mathbf{M}} Y$, and for each $Z \subseteq \overline{U}$,

> there exists $Y' \subsetneq Y$ such that $Y' \cup Z$ is a circuit if and only if there exists $X' \subsetneq X$ such that $X' \cup Z$ is a circuit.

It is easily observed that $\simeq^U_{\mathbf{M}}$ is an equivalence relation. It is not hard to check also that $X \simeq^U_{\mathbf{S}} Y$ whenever $X \equiv^U_{\mathbf{M}} Y$ and $\{[X']_{\equiv^U_{\mathbf{M}}} \mid X' \subsetneq X\} = \{[Y']_{\equiv^U_{\mathbf{M}}} \mid Y' \subsetneq Y\}$, *i.e.*, the number of equivalence classes of $\simeq^U_{\mathbf{M}}$ is bounded by $\mathsf{shadow}_{\mathbf{M}}(U) \cdot 2^{\mathsf{shadow}_{\mathbf{M}}(U)}$.

Proposition IV.5. *If a matroid $\mathbf{M}$ has linear decomposition-width k, then $\mathbf{M}$ is the value of a word $\mathbb{S}_1 \cdots \mathbb{S}_m$ in $\mathbf{BSS}_{2^k \times 2^{2^k}}$ such that, for every $1 \le i \le m$ and each interaction $\mathbf{i} \in \mathbf{I}^{\mathrm{R}}_i = \mathbf{I}^{\mathrm{L}}_{i+1}$, all the sets internally realising $\mathbf{i}$ in $\mathbb{S}_1 \odot \cdots \odot \mathbb{S}_i$ (resp. $\mathbb{S}_{i+1} \odot \cdots \odot \mathbb{S}_m$), are equivalent with respect to $\simeq^{S_{\le i}}_{\mathbf{S}}$ (resp. $\simeq^{S_{>i}}_{\mathbf{S}}$) where $S_{\le i}$ and $S_{>i}$ are, respectively, $S_1 \cup \cdots \cup S_i$ and $S_{i+1} \cup \cdots \cup S_m$.*

Remark IV.6. This technical proposition is needed when, given a bi-ported set system $\mathbb{S}$ admitting a binary Simon Factorisation $\mathbb{S}_1\mathbb{S}_2$, we want to compute $\mathbb{S}_1$ and $\mathbb{S}_2$. This property will allow to guess representatives and bind deterministically each set with its unique representative. It will be also crucial when the Simon factorisation of a bi-ported set system contains trivial hyperedges. Indeed, if $\mathbb{S}_1 \cdots \mathbb{S}_p$ is an unranked Simon factorisation of $\mathbb{S}$, then trivial hyperedges prevent from characterising $S \cap S_i$, for any $1 \le i \le p$. The fact that interactions have such a meaning on the circuits of the original matroid will however allow us to define an operation on bi-ported set systems so that we can reduce the case with trivial hyperedges to the case without trivial hyperedges.

B. Abstractions of bi-ported set systems

The construction of the transduction is by induction using Simon's Factorisation Forest Theorem, and for that we need to define a semi-group homomorphism from the set of words in $\mathbf{BSS}_k$ to a finite semi-group. As in [5], [2], the semi-group will describe some local connectivity allowing to deal with the unranked factorisation. Indeed, in the unranked factorisation, we need to colour the elements with a bounded number of colours and use this colouring to identify which elements of the domain belong to the same block of the factorisation. However, this is impossible in general - otherwise we might be able to MSO-transduce a linear ordering -, so we need to restrict the colouring to subsets of elements for which we can characterise when two elements in the domain belong to the same block. While in the graph case such subsets can be defined using connected components, there seems to be many obstacles in doing so for matroids or in general set systems. What we can first remark is that if $\mathbb{S}$ appears as an unranked factor in Simon's Factorisation of the word generating a matroid $\mathbf{M}$, then the set of circuits of $\mathbf{M}|_S$ appear as internal hyperedges in $\mathbb{S}$. It is attempting then to consider connected components of $\mathbf{M}|_S$ as subsets to colour. Unfortunately, even though we can show that whenever two elements of the same block are *internally connected*, then they are *internally connected* in a bounded window, we cannot obtain a characterization because of *trivial hyperedges* $((\mathsf{l}^{\mathrm{L}}, \mathsf{r}^{\mathrm{L}}), \emptyset, (\mathsf{l}^{\mathrm{R}}, \mathsf{r}^{\mathrm{R}}))$. In the graph case, this situation was solved in [2] by removing the vertices in common in the interactions. In our case, the interactions are not element matroids, so we need a more elaborate operation. We will however be able to overcome this difficulty by showing that we can expand such a bi-ported set system into a matroid and contract a subset that allows to reduce into the case without trivial hyperedges. This is done unfortunately at the price of a bit more complicated abstraction semi-group, and also more constraints on the bi-ported set system describing the input matroid, besides the more particular constraint that the input matroid belongs to a strongly-pigeonhole class. Let us now introduce some terminologies, before defining abstractions.

Let $\mathbb{S} = (\mathbf{I}^{\mathrm{L}}, S, \mathbf{I}^{\mathrm{R}}; \mathcal{S})$ be a bi-ported set system. Say that a subset Z of S *links* $\mathbf{i}_1 \in \mathbf{I}^{\mathrm{L}}$ and $\mathbf{i}_2 \in \mathbf{I}^{\mathrm{R}}$ when $(\mathbf{i}_1, Z, \mathbf{i}_2) \in \mathcal{S}$. When the empty set links some $\mathbf{i}_1 \in \mathbf{I}^{\mathrm{L}}$ to some $\mathbf{i}_2 \in \mathbf{I}^{\mathrm{R}}$ we say that they are *trivially linked*.

A bi-ported set system $\mathbb{S}$ is called a *circuit bi-ported set system* if for every $\mathbf{i} \in \mathbf{I}^{\mathrm{L}}$ and $\mathbf{j} \in \mathbf{I}^{\mathrm{R}}$, there are no X and $X' \subsetneq X$ such that both $(\mathbf{i}, X, \mathbf{j})$ and $(\mathbf{i}, X', \mathbf{j})$ are hyperedges. It is worth mentioning that if $\mathbb{S}_1$ and $\mathbb{S}_2$ are both circuit bi-ported set systems, then $\mathbb{S}_1 \odot \mathbb{S}_2$ is not necessarily a circuit bi-ported set system. However, if $\mathbb{S}_1 \odot \cdots \odot \mathbb{S}_m$ is a matroid, then each factor $\mathbb{S}_i \odot \cdots \odot \mathbb{S}_j$, for $1 \le i \le j \le m$, is a circuit bi-ported set system. Let's now introduce abstractions. The *abstraction of a circuit bi-ported set system* $\mathbb{S}$, denoted by $[\mathbb{S}]$, is the tuple $(\mathbf{I}^{\mathrm{L}}, \mathbf{I}^{\mathrm{R}}; \mathbf{AL}; \mathbf{ASL})$ where

1) $\mathbf{AL}$ is the edge-labeled bipartite graph between $\mathbf{I}^{\mathrm{L}}$ and $\mathbf{I}^{\mathrm{R}}$ with two vertices adjacent when there is some subset Z of S that links these label interactions, and the label of an edge $(\mathbf{i}, \mathbf{j})$ is 0 if $\mathbf{i}$ and $\mathbf{j}$ are trivially linked, and 1 otherwise. We call it the *link graph*.
2) $\mathbf{ASL}$ is the edge-labeled directed graph on vertex-set $\mathbf{I}^{\mathrm{L}} \times \mathbf{I}^{\mathrm{R}}$, and there is an arc $((\mathbf{i}_1, \mathbf{j}_1), (\mathbf{i}_2, \mathbf{j}_2))$ if there are $X \subseteq S$ and a subset X' of X such that $(\mathbf{i}_1, X, \mathbf{j}_1)$ and $(\mathbf{i}_2, X', \mathbf{j}_2)$ are both hyperedges in $\mathcal{S}$; each edge is labeled by 1 (there are X and $X' \subsetneq X$) or 0 ($X' = X$ and no $X' \subsetneq X$). We call it the *strong link graph*.

Notice that $\mathbf{i} \in \mathbf{I}^{\mathrm{L}}$ and $\mathbf{j} \in \mathbf{I}^{\mathrm{R}}$ cannot be linked at the same time by both X and $X' \subsetneq X$ in circuit bi-ported set systems, which explains why the labelings of $\mathbf{AL}$ and of $\mathbf{ASL}$ are both exclusive. Let $[\mathbf{0}]$ be a new element, that can be considered as denoting "syntactical error". The *abstraction of any bi-ported set system* that is not a circuit bi-ported set system is defined as $[\mathbf{0}]$. Let $\mathbf{A}_k$ be the set that contains all the possible abstractions of circuit bi-ported set systems using labels $\mathrm{L} = \mathrm{R} = [k]$, as well as the special abstraction $[\mathbf{0}]$. The following summarises the properties we are interested on abstractions, in particular it admits a semi-group structure.

Proposition IV.7. *The set $\mathbf{A}_k$ has size at most $3^{k^4(k^4+1)}$. There is moreover an associative operation $[\tilde{\odot}]$ such that $[\mathbb{S}_1 \odot \mathbb{S}_2] = [\mathbb{S}_1]\,[\tilde{\odot}]\,[\mathbb{S}_2]$.*

C. Proof of definability

We prove here Theorem III.2. For every positive integer k, let $\approx_k : \mathbf{BSS}^+_k \to \mathbf{A}_k$ be the function that maps every word $\mathbb{S}_1\mathbb{S}_2 \cdots \mathbb{S}_m$ into $[\mathbb{S}_1 \odot \mathbb{S}_2 \odot \cdots \odot \mathbb{S}_m]$. We have seen in Subsection IV-B that $\approx_k$ is a semi-group homomorphism. The proof will be by induction on the $\approx_k$-rank of words in $\mathbf{BSS}^+_k$ defining matroids and satisfying the conditions of Proposition IV.5.

Throughout the section, we let f be a computable function and we are assuming that every correct input $\mathbb{S}$ to the transduction satisfies the following: There are bi-ported set systems $\mathbb{A}$ and $\mathbb{Z}$ such that $\mathbb{A} \odot \mathbb{S} \odot \mathbb{Z}$ is a matroid $\mathbf{M}$, and $\mathsf{shadow}_{\mathbf{M}}(X) \leq f(\lambda_{\mathbf{M}}(X))$ for every $X \subseteq M$. We call such bi-ported set systems *matroidal bi-ported set systems*, and $\mathbb{A}$ and $\mathbb{Z}$ its *certificates*. We recall that if w is a word in $\mathbf{BSS}_k$ defining a matroid $\mathbf{M}$ with $\mathsf{shadow}_{\mathbf{M}}(X) \leq f(\lambda_{\mathbf{M}}(X))$, then any factor of w has as value a matroidal bi-ported set system. We let *g be the function* where $g(1,t) = f(t)$ for every integer $t \geq 0$, and $g(\ell, t) = f(g(\ell-1, t) + 3t)$ for all integers $\ell \geq 2$ and $t \geq 0$.

A bi-ported set system is viewed as an extended relational structure in which the universe consists in the ground set elements together with the left and right interactions, and there are two monadic relations Left and Right which correspond, respectively, to left and right interactions, and two predicates $\mathsf{lhyperedge}$ and $\mathsf{rhyperedge}$ which, respectively, select tuples $(\{\mathsf{l}^{\mathrm{L}}\}, X, \{\mathsf{l}^{\mathrm{R}}\})$ and $(\{\mathsf{r}^{\mathrm{L}}\}, X, \{\mathsf{r}^{\mathrm{R}}\})$ such that $((\mathsf{l}^{\mathrm{L}}, \mathsf{r}^{\mathrm{L}}), X, (\mathsf{l}^{\mathrm{R}}, \mathsf{r}^{\mathrm{R}}))$ is a hyperedge.

Definition IV.8. *Let $\mathbb{S}$ be a matroidal bi-ported set system that is the value of a word $\mathbb{S}_1 \cdots \mathbb{S}_m \in \mathbf{BSS}_k^+$. The definable decomposition-width of $\mathbb{S}$, denoted by $\mathsf{ddw}(\mathbb{S})$, is the minimum quantifier-height of an MSO-transduction computing a rooted layout of $\mathbb{S}$ of $\mathsf{shadow}_{\mathbb{S}}$-width at most $g(\ell, k)$, with ℓ the $\approx_k$-rank of $\mathbb{S}_1 \cdots \mathbb{S}_m$.*

We are going to prove the following, which combined with Simon's Factorisation Theorem and Proposition IV.7 and Theorem III.2.

Proposition IV.9. *For every positive integer k, there is a function f_k such that*

$$\mathsf{ddw}(\mathbb{S}_1 \odot \cdots \odot \mathbb{S}_m) \leq f_k(\max\{\mathsf{ddw}(\mathbb{S}_i) \mid 1 \leq i \leq m\}),$$

where $\mathbb{S}_1 \cdots \mathbb{S}_m$ is an $\approx_k$-factorisation of a matroidal bi-ported set system $\mathbb{S}$.

We consider two cases in proving Proposition IV.9 : either the $\approx_k$-factorisation is binary or it is unranked, and in the latter case we need to consider the cases with and without trivial hyperedges. In the remaining, let $\mathbb{S}$ be a matroidal bi-ported set system and let also $\mathbb{A}$ and $\mathbb{Z}$ be certificates of $\mathbb{S}$ such that $\mathbb{A} \odot \mathbb{S} \odot \mathbb{Z} = \mathbf{M}$.

1) Binary case: Let $\mathbb{S}_1\mathbb{S}_2$ be an $\approx_k$-factorisation of $\mathbb{S}$. Assume that there are MSO-transductions τ_1 and τ_2, of quantifier-heights at most q, that on inputs, respectively, $\mathbb{S}_1$ and $\mathbb{S}_2$, output, respectively, a rooted layout of width at most c. We recall that if $\mathbb{S} = \mathbb{S}_1 \odot \mathbb{S}_2$, then $\mathbf{I}_1^{\mathrm{R}} = \mathbf{I}_2^{\mathrm{L}}$, $\mathbf{I}^{\mathrm{L}} = \mathbf{I}_1^{\mathrm{L}}$ and $\mathbf{I}^{\mathrm{R}} = \mathbf{I}_2^{\mathrm{R}}$, and also $\mathbf{I}^{\mathrm{L}}, \mathbf{I}^{\mathrm{R}}, \mathbf{I}_1^2 \subseteq \mathrm{L} \times \mathrm{R}$.

Let's make some observations before describing the MSO-transduction. For $\mathbf{i} \in \mathbf{I}^{\mathrm{L}}$ and $X_1, X_1' \subseteq S_1$, we write $X_1 \sim_{\mathbf{i}}^{\mathrm{L}} X_1'$ if, for any $\mathbf{j} \in \mathbf{I}^{\mathrm{R}}$ and $X_2 \subseteq S_2$, we have $(\mathbf{i}, X_1 \cup X_2, \mathbf{j}) \in \mathcal{S}$ if and only if $(\mathbf{i}, X_1' \cup X_2, \mathbf{j}) \in \mathcal{S}$. First observe that $\sim_{\mathbf{i}}^{\mathrm{L}}$ is an equivalence relation and has at most $|\mathbf{I}_1^{\mathrm{R}}|$ equivalence classes. Moreover, if $\mathbb{A} \odot \mathbb{S} \odot \mathbb{Z}$ is a matroid $\mathbf{M}$, then for any $U \subseteq A$ and $X_1, X_1' \subseteq S_1$, we have that $U \cup X_1 \simeq_{\mathbf{M}}^{A \cup S_1} U \cup X_1'$ whenever $((\mathsf{l}^{\emptyset}, \cdot), U, \mathbf{i}) \in \mathcal{A}$, and $\{[Y]_{\sim_{\mathbf{i}'}^{\mathrm{L}}} \mid \mathbf{i}' \in \mathbf{I}^{\mathrm{L}},\ Y \subseteq X_1\} = \{[Y]_{\sim_{\mathbf{i}'}^{\mathrm{L}}} \mid \mathbf{i}' \in \mathbf{I}^{\mathrm{L}},\ Y \subseteq X_1'\}$. Therefore, let's write $X_1 \simeq_{\mathbf{i}}^{\mathrm{L}} X_1'$, for $X_1, X_1' \subseteq S_1$, whenever $X_1 \sim_{\mathbf{i}}^{\mathrm{L}} X_1'$ and $\{[Y]_{\sim_{\mathbf{i}'}^{\mathrm{L}}} \mid \mathbf{i}' \in \mathbf{I}^{\mathrm{L}},\ Y \subseteq X_1\} = \{[Y]_{\sim_{\mathbf{i}'}^{\mathrm{L}}} \mid \mathbf{i}' \in \mathbf{I}^{\mathrm{L}},\ Y \subseteq X_1'\}$, which is an equivalence relation. Therefore, $\simeq_{\mathbf{i}}^{\mathrm{L}}$ is a refinement of both $\simeq_{\mathbf{M}}^{A \cup S_1}$ and of $\equiv_{\mathbf{M}}^{A \cup S_1}$, and has at most $|\mathbf{I}_1^{\mathrm{R}}| \cdot 2^{|\mathbf{I}^{\mathrm{L}}| \cdot |\mathbf{I}_1^{\mathrm{R}}|}$ equivalence classes, that we denote by k_i^{L}. We define similarly, equivalence relations $\sim_{\mathbf{j}}^{\mathrm{R}}$ and $\simeq_{\mathbf{j}}^{\mathrm{R}}$, for $\mathbf{j} \in \mathbf{I}^{\mathrm{R}}$, on subsets of S_2 and denote the number of equivalence classes of $\sim_{\mathbf{j}}^{\mathrm{R}}$ by $k_{\mathbf{j}}^{\mathrm{R}}$.

Let's describe the transductions, whose composition, computes a rooted layout of $\mathsf{shadow}_{\mathbb{S}}$-width at most $f(c+2k)$ of $\mathbb{S}$. We omit details of the transductions for page constraints. While we need to be careful, there are no difficulties in producing them from the description given below.

1) The first one colors two subsets S_1 and S_2 forming a partition of S.
2) The second is the composition of the following, constructed for each $\mathbf{i} \in \mathbf{I}^{\mathrm{L}}$. For each $\mathbf{i} \in \mathbf{I}^{\mathrm{L}}$, we have an MSO-transduction that colors $k_{\mathbf{i}}^{\mathrm{L}}$ subsets of S_1 and checks that they are representatives of the equivalence classes of $\simeq_{\mathbf{i}}^{\mathrm{L}}$. It also creates an extended relation rep_1 where $\mathsf{rep}_1(\mathbf{i}, X, X_t)$ holds if and only if $X \simeq_{\mathbf{i}}^{\mathrm{L}} X_t$ and X_t is among the colored set. Notice that because $\simeq_{\mathbf{i}}^{\mathrm{L}}$ is an equivalence relation, for each $X \subseteq S_1$, there is exactly one X_t such that $X \simeq_{\mathbf{i}}^{\mathrm{L}} X_t$.
3) The third is exactly the same as the second transduction, except it computes $\mathsf{rep}_2(\mathbf{j}, Y, Y_t)$ for $\mathbf{j} \in \mathbf{I}^{\mathrm{R}}$ and $Y \subseteq S_2$ and colors $k_{\mathbf{j}}^{\mathrm{R}}$ subsets of S_2, which are intended to be representatives of the equivalence classes of $\simeq_{\mathbf{j}}^{\mathrm{R}}$.
4) The fourth transduction will compute bi-ported set systems $\mathbb{S}_1$ and $\mathbb{S}_2$, on ground sets, respectively, S_1 and S_2 and such that $\mathbb{S} = \mathbb{S}_1 \odot \mathbb{S}_2$. This step can be done by a sequence of MSO-transductions using relations rep_1 and rep_2.
   We have thus proved that if $\mathbb{S} = \mathbb{S}_1 \odot \mathbb{S}_2$, then there is an MSO-transduction that guesses $\mathbb{S}_1$ and $\mathbb{S}_2$. Let's now combine this MSO-transduction with τ_1 and τ_2 to compute a rooted layout of $\mathbb{S}$.
5) The last MSO-transduction consists in first running τ_1 on $\mathbb{S}_1$ and then τ_2 on $\mathbb{S}_2$ and keeps only $\mathbb{S}$, the rooted layouts (T_1, δ_1) of S_1 and (T_2, δ_2) of S_2 and a bijective mapping between S and the leaves of $\mathsf{T}_1 \cup \mathsf{T}_2$, which can be constructed from (T_1, δ_1) and (T_2, δ_2). Then, it colors one element of S and creates a rooted layout rooted at a copy of the colored element and whose children are the roots of T_1 and T_2. It finally cleans everything, except $\mathbb{S}$ and the rooted layout of $\mathbb{S}$.

It is routine now to check that each step is an MSO-transduction of quantifier-height depending only on $|\mathbf{I}^{\mathrm{L}}|, |\mathbf{I}^{\mathrm{R}}|$ and $|\mathbf{I}_1^{\mathrm{R}}| = |\mathbf{I}_2^{\mathrm{L}}|$. Since, we combine a constant number of MSO-transductions, each of quantifier-height depending only on $|\mathbf{I}^{\mathrm{L}}|, |\mathbf{I}^{\mathrm{R}}|, |\mathbf{I}_1^{\mathrm{R}}| = |\mathbf{I}_2^{\mathrm{L}}|$, and q, the computed MSO-transduction has quantifier-height at most $f_k(q)$ for some function f_k, by [10, Theorems 7.10 and 7.14]. The fact that the computed rooted layout can be assumed to have $\mathsf{shadow}_{\mathbb{S}}$-width at most $f(c+2k)$ follows from Lemma II.3 combined with the fact that τ_1 (resp. τ_2) compute rooted layouts of $\mathsf{shadow}_{\mathbb{S}_1}$-width (resp. $\mathsf{shadow}_{\mathbb{S}_1}$-width) bounded by c, and the fact that $\lambda_{\mathbf{M}}(S_1)$ and $\lambda_{\mathbf{M}}(S_2)$ are both bounded by $2k$.

2) Unranked case: Let $\mathbf{0} \neq \mathbb{S} = \mathbb{S}_1 \odot \cdots \odot \mathbb{S}_p$ with $[\mathbb{S}_1] = \cdots = [\mathbb{S}_p] = (\mathbf{I}^{\mathrm{L}}, \mathbf{I}^{\mathrm{R}}; \mathbf{AL}; \mathbf{ASL})$. Because $[\mathbb{S}_1]\,[\tilde{\odot}]\,[\mathbb{S}_2] = [\mathbb{S}_1 \odot \mathbb{S}_2]$ and $[\mathbb{S}_1] = \cdots = [\mathbb{S}_p]$, we can conclude that $\mathbf{I}^{\mathrm{L}} = \mathbf{I}^{\mathrm{R}} = \mathbf{I}$, *i.e.*, $\mathbb{S}_n = (\mathbf{I}, S_n, \mathbf{I}; \mathcal{S}_n)$, for all $1 \leq n \leq p$. Recall that $\mathsf{l}^{\emptyset}$ and $\mathsf{r}^{\emptyset}$ are reserved for the equivalence class of the subset $\emptyset$ as we are dealing with matroids. Since moreover, the bi-ported set system we are manipulating will produce a matroid, the $\emptyset$ cannot be a resultant set.

Unranked case without trivial hyperedges. We are not going to use the idempotency neither the strong link graph here, but we will do in the case where trivial hyperedges exist. We will show that whenever we restrict ourselves to an *internal connected component* C, then we are able to guess $C \cap S_n$, for each $1 \leq n \leq p$, and the linear ordering of the S_i's intersecting C. In a second step, we prove that one can combine the rooted layouts of small width of the internal connected components into one of small width of S. Say that x and y in S are *internally connected* if there is an internal hyperedge X of $\mathcal{S}$ such that $x, y \in X$. An *internal connected component* of $\mathbb{S}$, is a maximal subset C of S such that any pair x, y of C are internally connected. The following characterises internal connected components of matroidal bi-ported set systems.

Lemma IV.10. *Let $\mathbb{S}$ be a matroidal bi-ported system. Then "being internally connected" is an equivalence relation and each internal connected component corresponds to an equivalence class of it.*

The following says that a matroidal bi-ported set system always has a rooted layout of small $\mathsf{shadow}_{\mathbb{S}}$-width displaying internal connected components.

Lemma IV.11. *Let $\mathbb{S}$ be a matroidal bi-ported set system. If the decomposition-width of $\mathbb{S}$ is k, then $\mathbb{S}$ has a rooted-layout (T, δ) of $\mathsf{shadow}_{\mathbb{S}}$-width k such that each subtree rooted at a child of the root corresponds to a rooted layout of one internal connected component of $\mathbb{S}$.*

The goal now is to prove that we can color the elements of S with a bounded number of colours and using this colouring we can identify $C \cap S_n$ for each internal connected component. Let $\mathsf{bc} : [\![1, p]\!] \to [\![0, 2|\mathbf{I}| + 5]\!]$ be the mapping where $\mathsf{bc}(1) = 0$ and, for each $2 \leq n \leq p$,

$$\mathsf{bc}(n) = (\mathsf{bc}(n-1) + 1) \mod (2|\mathbf{I}| + 6).$$

Notice that, for every $n \geq |\mathbf{I}| + 3$, there is a unique $r \in [\![0, 2|\mathbf{I}| + 5]\!]$ such that $r \notin \{\mathsf{bc}(m) \mid m \in [\![n - (|\mathbf{I}| + 2), n + (|\mathbf{I}| + 2)]\!]\}$, and let's denote this unique integer r by $-\mathsf{bc}(n)$.

Let's now define the mapping $\mathsf{c} : S \to [\![0, 2|\mathbf{I}| + 5]\!]$ where, for each $1 \leq n \leq p$ and each $x \in S_n$, we have $\mathsf{c}(x) = \mathsf{bc}(n)$. It is worth mentionning that c induces a partition of S into $2|\mathbf{I}| + 6$ parts, by colouring the sequence $S_1, S_2, \ldots, S_p$ in a cyclic ordering $0, 1, 2, \ldots, 2|\mathbf{I}| + 5, 0, 1, 2, \ldots, 2|\mathbf{I}| + 5, \ldots$.

Lemma IV.12. *Let x and y be two elements in the same internal connected component of $\mathbb{S}$. Then, there is $1 \leq n \leq p$ such that x and y belong to S_n if and only if $\mathsf{c}(x) = \mathsf{c}(y)$ and x and y are in an internal hyperedge X not intersecting S_m, for any $1 \leq m \leq p$ with $\mathsf{bc}(m) = -\mathsf{bc}(n)$.*

In the same way, we can prove the following.

Lemma IV.13. *Let x and y be two elements in the same internal connected component of $\mathbb{S}$. Then, there is $1 \leq n \leq p$ such that $x \in S_n$ and $y \in S_{n+1}$ if and only if $\mathsf{c}(y) = (\mathsf{c}(x) + 1) \mod (2|\mathbf{I}|+6)$, and x and y are in an internal hyperedge X not intersecting S_m, for any $1 \leq m \leq p$ with $\mathsf{bc}(m) = -\mathsf{bc}(n+1)$.*

We are now ready to give the MSO-transduction that computes a rooted layout of $\mathbb{S}$ from the MSO-transduction τ that takes as input a bi-ported set system $\mathbb{S}_n$, for $1 \leq n \leq p$, and outputs a rooted layout of $\mathbb{S}_n$. But, before we need a last technical lemma. Indeed, what we are able to characterise is $C \cap S_n$, for each internal connected component C of $\mathbb{S}$, while the transduction τ is taken as input $\mathbb{S}_n$. We prove in the next lemma that there exists an MSO-transduction τ' that, given the internal hyperedges of $\mathcal{S}$ included in $C \cap S_n$, extend it into a bi-ported set system $\mathbb{S}'_n$ such that $\tau(\mathbb{S}_n)$ produces an output if and only if $\tau'(\mathbb{S}'_n)$ does.

Lemma IV.14. *Let τ be an effectively given MSO-transduction that takes as input matroidal bi-ported set systems $\mathbb{S}$ having interactions in $\mathbf{I}$ and certificates $\mathbb{A}$ and $\mathbb{Z}$ such that $\mathbb{A} \odot \mathbb{S} \odot \mathbb{Z}$ has decomposition-width q. For every positive integer k, there exists an effectively given MSO-transduction τ_k, taking as inputs k-ported set systems and such that, for every matroidal bi-ported set system $\mathbb{S}$ and k-ported set system $(\mathbf{N}, \rho)$ with non-emptyset $N \subseteq S$ and $\mathsf{shadow}_{\mathbb{S}}(N) = k$,*

$$\tau(\mathbb{S}) \text{ has an output if and only if } \tau_k(\mathbf{N}, \rho) \text{ has an output.}$$

The MSO-transduction for an internal connected component C consists then in guessing the colouring described above and define the equivalence relation $\mathsf{same{-}bag}$ stating that two elements in the internal connected component belong to the same $\mathbb{S}_i$ (this is possible thanks to Lemma IV.12). Then, it calls in parallel the transduction from Lemma IV.14 in all the equivalence classes of $\mathsf{same{-}bag}$, which is possible by Lemma II.5 and the fact that Lemma IV.12 gives an equivalence relation. Using finally the characterisation from Lemma IV.13 one can construct an MSO-transduction that takes as input all the rooted layouts of equivalence classes of $\mathsf{same{-}bag}$ and compute a rooted layout of C of desired $\mathsf{shadow}_{\mathbb{S}}$-width.

The MSO-transduction for $\mathbb{S}$ consists then in calling first, in parallel, the transduction for connected components, in all internal connected components (recall it is possible thanks to Lemma II.5 and "being internally connected" is an equivalence relation). Then it constructs a rooted layout where each child of the root is the root of a rooted layout of some internal connected component. Again we omit the technical details that can be found in the appendix due to page constraints.

Unranked case with trivial hyperedges. Contrary to the previous case where neither the idempotency nor the strong link graph are needed, both are needed here to prove that we can reduce to the case without trivial hyperedges. The transduction will be the composition of two: one that extends $\mathbb{S}$ into a matroid, and one that uses this last matroid to compute a bi-ported set system admitting an $\approx_k$-factorisation without trivial hyperedges, so that we can call the previous MSO-transduction.

Suppose that $\mathbf{0} \neq \mathbb{S} = \mathbb{S}_1 \odot \cdots \odot \mathbb{S}_p$ with $[\mathbb{S}] = [\mathbb{S}_1] = \cdots = [\mathbb{S}_p] = (\mathbf{I}^{\mathrm{L}}, \mathbf{I}^{\mathrm{R}}; \mathbf{AL}; \mathbf{ASL})$ is idempotent. We remind that $\mathbb{S}$ is matroidal and let $\mathbb{A}$ and $\mathbb{Z}$ be certificates such that $\mathbb{A} \odot \mathbb{S} \odot \mathbb{Z} = \mathbf{M}$. By the assumption also on the word defining $\mathcal{M}$ (see the equivalence relation $\simeq^U_{\mathbf{M}}$ for each subset U of M and the definition of bi-ported set systems), we know that only the $\emptyset$ is equivalent to $\mathsf{l}^\emptyset$ and $\mathsf{r}^\emptyset$. In particular, whenever $(\mathsf{l}^\emptyset \neq \mathsf{l}^{\mathrm{L}}, \cdot)$ (resp. $(\cdot, \mathsf{r}^{\mathrm{R}} \neq \mathsf{r}^\emptyset)$) belongs to $\mathbf{I}^{\mathrm{L}}$, then there is a non-empty $X \subseteq A$ such that $[X]_{\simeq^A_{\mathbf{M}}}$ is l^{L} (resp. non-empty $X \subseteq B$ such that $[X]_{\simeq^B_{\mathbf{M}}}$ is r^{R}). By using this, we first prove that after contracting B, there are no more trivial hyperedges, when we restrict to S. Then, we show that, the bi-ported set system reduced to $S \setminus (S_1 \cup S_p)$, after contracting B, has an $\approx_k$-factorisation allowing to use the previous MSO-transduction. From now on, let $\mathbf{N}$ be the matroid $(\mathbf{M}/B)|_S$.

Lemma IV.15. *Let $1 \leq n < m \leq p$. Let $x \in S_n$ and $y \in S_m$. There is no circuit $C \subseteq S_n \cup \cdots \cup S_m$ of $\mathbf{N}$ containing both x and y such that $C \cap S_a = \emptyset$ for some $n+1 \leq a \leq m-1$.*

By Lemma IV.15, for each circuit C of N, there is an interval $[\![n, m]\!] \subseteq [p]$ such that $C \subseteq S_n \cup \cdots \cup S_m$ and $C \cap S_a \neq \emptyset$ for all $a \in [\![n, m]\!]$. Again, there are bi-ported set systems $\mathbb{N}_1, \ldots, \mathbb{N}_p$ on, respectively, $S_1, \ldots, S_p$, such that the set of circuits of N is exactly the set system described by $\mathbb{N} = \mathbb{N}_1 \odot \cdots \odot \mathbb{N}_p$, and no internal hyperedge in $\mathbb{N}_n \odot \cdots \odot \mathbb{N}_m$, for $1 \leq n \leq m \leq p$, uses trivial hyperedges (by Lemma IV.15). We are going to prove that the link graph in $[\mathbb{N}_a]$ is the same as the link graph in $[\mathbb{N}_2]$, for all $3 \leq a \leq p-1$, which will be sufficient to reduce to the case where no internal hyperedge goes through trivial hyperedges. Our main obstacle is the description of interactions in $\mathbb{N}$, and we will overcome this difficulty by showing that we can in fact keep the same as in $\mathbb{S}$. We will do it in two steps. First, we prove that if X and Y, subsets of $N_a = S_a$, are $\simeq^{S_{\leq a}}_{\mathbf{M}}$-equivalent, then they remain $\simeq^{S_{\leq a}}_{\mathbf{N}}$-equivalent.

Lemma IV.16. *Let $1 \leq n \leq p$ and let $\mathbf{i} \in \mathbf{I}^{\mathrm{R}}$ be internally realised in $\mathbb{A} \odot \mathbb{S}_1 \odot \cdots \odot \mathbb{S}_n$ by X and Y, both subsets of $S_{\leq i}$. Then, $X \simeq^{S_{\leq n}}_{\mathbf{N}} Y$.*

In a second step, we show that $\{\mathbf{i}_1, \mathbf{i}_2\}$ exists in the link graph of $[\mathbb{N}_a]$, for $2 \leq a \leq p-1$, if and only if it exists in the link graph of $[\mathbb{N}_2]$. As a consequence, in order to describe $[\mathbb{N}_a]$, we can use the same interactions as in $\mathbb{S}_a$. Also, we are only interested in internally realised interactions and we can consider only them when computing $\mathbb{N}$. Because $[\mathbb{S}] = [\mathbb{S}_1] = \cdots = [\mathbb{S}_p]$ is idempotent, we know that $\mathbf{i}$ is internally realised in $\mathbb{S}$ if and only if it is internally realised in each $\mathbb{S}_i$, and because $\mathbf{I}^{\mathrm{L}} = \mathbf{I}^{\mathrm{R}}$, we do not lose any generality by restricting the interactions of each $\mathbb{N}_a$ to the internally realised ones. The following is then a corollary of Lemma IV.16.

Lemma IV.17. *Let $2 \leq a \leq p$ and let $\mathbf{i}$ and $\mathbf{j}$ be two internally realised interactions in $\mathbb{S}$. For every two subsets X and Y of S_a such that $(\mathbf{i}, X, \mathbf{j})$ and $(\mathbf{i}, Y, \mathbf{j})$ are hyperedges in $\mathcal{S}_a$, the following are equivalent:*

1) *$(\mathbf{i}, X, \mathbf{j})$ is a hyperedge in $\mathbb{N}_a$.*
2) *$(\mathbf{i}, Y, \mathbf{j})$ is a hyperedge in $\mathbb{N}_a$.*

Therefore, for any two internally realised interactions $\mathbf{i}$ and $\mathbf{j}$, the question whether there is X such that $(\mathbf{i}, X, \mathbf{j})$ is a hyperedge in $\mathcal{N}_a$ can be answered using the abstraction of $\mathbb{S}$. Notice that the link graph in $[\mathbb{N}_p]$ may be distinct from the link graph in $[\mathbb{N}_2]$ because $\mathbb{N}_p$ has no right interaction (except the trivial ones $(\cdot, \mathsf{r}^\emptyset)$); and similarly $\mathbb{N}_1$ has no left interaction (except the trivial ones $(\mathsf{l}^\emptyset, \cdot)$) and then the link graph in $[\mathbb{N}_1]$ may be distinct from the link graph in $[\mathbb{N}_2]$. The following summarises the properties.

Proposition IV.18. *For every $2 \leq a \leq p-1$, there is no trivial hyperedge in $\mathcal{N}_a$ and moreover, $\mathbf{AL}_{\mathbb{N}_a} = \mathbf{AL}_{\mathbb{N}_2}$.*

We are now ready to describe the MSO-transduction, as before we give the main steps, each being clearly a transduction which quantifier-height depends only on the number of interactions and the quantifier-height of the MSO-transduction described when there are no trivial hyperedges. First, recall there is an MSO-transduction χ of constant quantifier-height, which given a matroid M and a tri-partition (A, S, B) of its ground set, computes the matroid $\mathbf{M}/B|_S$. One can use then pumping arguments and the strongly pigeonhole property to prove that there is an MSO-transduction χ' of quantifier-height $f(|\mathbf{I}|)$, for some computable function f depending on χ, which takes as input a bi-ported set system $\mathbb{S}$ over interactions $\mathbf{I}$ for which there are $\mathbb{A}$ and $\mathbb{B}$ with $\mathbb{A} \odot \mathbb{S} \odot \mathbb{B}$ a matroid M of decomposition-width k and computes $(\mathbf{M}/B)|_S$. Since the MSO-transduction χ' can first guess representatives of $\mathbb{A}$ and $\mathbb{Z}$, we can also use χ' to guess S_1, S_p and compute at the same time (by making some copies first if necessary) the bi-ported set systems $\mathbb{S}_1, \mathbb{S}_p, \mathbb{N}_1, \mathbb{N}_p$ and $\mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1}$ such that $\mathbf{N}$ is $\mathbb{N}_1 \odot \mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1} \odot \mathbb{N}_p$. This MSO-transduction can be computed in the same way as it was done in the binary case, and has quantifier-height depending only on k and the quantifier-height of χ. Here are now the different steps of the desired MSO-transduction:

1) The first step consists in running χ' on $\mathbb{S}$ to obtain the desired bi-ported set systems, and clear it by removing $\mathbb{N}_1$ and $\mathbb{N}_p$.
2) Because $\mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1}$ does not have trivial hyperedges and is an idempotent factorisation, we can run in a third step the MSO-transduction in the case without trivial hyperedges on $\mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1}$ and obtain a rooted layout of $\mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1}$.

3) Let τ be the MSO-transduction which takes as input some $\mathbb{S}_a$, for $1 \le a \le p$, and computes a rooted layout of S_a of $\mathsf{shadow}_{\mathbb{S}_a}$-width at most $g(\ell-1,k)$. The fourth step consists in calling τ, on $\mathbb{S}_1$ and then on $\mathbb{S}_p$.
4) The fifth step consists then in creating a rooted layout with root root whose first child is the root of the rooted layout of $\mathbb{S}_1$, its second child is root_1 and this latter has as first child the root of the rooted layout of $\mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1}$ and its second child is the root of the rooted layout of $\mathbb{S}_p$.
5) The last step consists in cleaning everything, except keeping the rooted layout of $\mathbb{S}$ (with $\mathbb{S}$ itself).

Let us now analyse the $\lambda_{\mathbf{M}|_S}$-width of the computed rooted layout. Let (T_1,δ_1), (T_2,δ_2) and (T_3,δ_3) be the rooted layouts of, respectively, $\mathbb{S}_1$, $\mathbb{N}_2 \odot \cdots \odot \mathbb{N}_{p-1}$ and of $\mathbb{S}_p$. It is easy to check that we need only to look at cuts in $\mathsf{T}_1, \mathsf{T}_2$ and T_3 in order to upper-bound the $\lambda_{\mathbf{M}|_S}$-width of the computed rooted layout. By inductive hypothesis and Proposition II.2, each cut in T_1 (resp. in T_2 and T_3) has $\lambda_{\mathbf{N}|_{S_1}}$-width (resp. $\lambda_{\mathbf{N}|_{S_p}}$-width and $\lambda_{\mathbf{N}|_{N\setminus(S_1\cup S_p)}}$-width) at most $g(\ell-1,k)$ and by Lemma II.3 has $\lambda_{\mathbf{N}|_S}$-width at most $g(\ell-1,k)+2k$. By Lemma II.4 and the fact that $\mathbf{N}$ is obtained by contracting B and $\lambda_{\mathbf{M}}(B) \le k$, we can conclude that each cut in T has $\lambda_{\mathbf{M}|_S}$-width at most $g(\ell-1,k)+3k$, *i.e.*, has $\mathsf{shadow}_{\mathbb{S}}$-width at most $f(g(\ell-1,k)+3k) = g(\ell,k)$.

Let's conclude with the quantifier-height of the MSO-transduction described above. Since each step uses an MSO-transduction whose quantifier-height and number of variables depend only on k and of the quantifier-height and number of variables of the MSO-transduction τ, we can conclude that the quantifier-height of the given MSO-transduction depend only on k and the definable decomposition-width of each $\mathbb{S}_a$, for $1 \le a \le p$, by [10, Theorems 7.10 and 7.14].

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