On the denotation of circular and non-wellfounded proofs in linear logic with fixed points

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***Abstract*—This paper investigates the denotational invariants of non-wellfounded and circular proofs of linear logic with least and greatest fixed points, $\mu\mathsf{LL}$, by providing a categorical semantics. More precisely the paper successively introduces semantics for (i) non-wellfounded pre-proofs, be they valid or not, (ii) valid pre-proofs exploiting their validity condition by considering an orthogonality construction on the given categorical model and finally (iii) circular strongly valid pre-proofs, exploiting both validity and regularity in order to define inductively the interpretation. Then the paper investigates the semantical content of the translation from finitary proofs to non-wellfounded proofs and, conversely, from (strongly valid) circular proofs to finitary proofs, showing that both translations preserve the interpretation.**

I. Introduction

a) Fixed-points in formulas and proofs: In the framework of logics with (co)induction (such as the μ-calculus, logics with inductive definitions, Kleene algebras, etc...), circular and non-wellfounded proofs (allowing infinitely long branches) have gained growing attention over the past twenty years. Different proof systems have been considered for various logics such as classical [11]–[13], intuitionistic [17] or linear logic [6], [23], [28], [44], as well as linear-time or branching-time temporal logics [1], [21], [24], [36], [57], and session-typed processes [22], [55].

Beside non-wellfounded proof systems, there are also finitary proof systems that allow us to do (co)inductive reasoning. For instance, in the case of linear logic, Baelde and Miller considered $\mu\mathsf{MALL}$, an extension of multiplicative additive linear logic with induction and coinduction principles [2], [7] in the form of Park's rules. Such finitary proof systems usually predate the circular ones. It seems accepted that if we want to have a cut-elimination theorem for the finitary proof systems with an induction principle, then the price to pay is to lose the sub-formula property [41]. There are basically two ways to solve this, by considering either infinitary logic in the sense of [50], [53], or non-wellfounded proofs as above.

In non-wellfounded proofs, derivation trees are not restricted to be finite trees anymore and this makes the logic *a priori* inconsistent (See Figure 4b for an example illustrating how one can derive any sequent): we refer to any arbitrary non-wellfounded derivation tree as a pre-proof. A syntactic notion of validity is then introduced to restore the consistency of the logic, and derivation trees that satisfy this validity are called valid pre-proofs, or simply proofs. Despite the fact that fixed-point inference rules are much simpler compared to finitary proof systems, the logic remains very expressive, even when restricted to the regular fragment, that is constraining derivations to be infinite *regular* trees, that we shall refer to as circular proofs. Such circular proofs can be represented as finite trees with back-edges (see an example below) which can be seen as a trivial fixed-point problem on proofs.

b) Relations between finitary and non-wellfounded proofs: The relationship between finitary and non-wellfounded proof systems is an important and, often, difficult question remaining open in a number of cases. In particular, in the sub-structural versions of the μ-calculus, it is not known whether the regular fragment of non-wellfounded proofs, coincides with the finitary fragment. For systems with Martin-Löf's inductive predicates [41], Berardi and Tatsuta showed [10] that circular and inductive proofs are not equivalent in general [13]. On the other hand, Simpson [51] and Berardi and Tatsuta [9] showed that circular and inductive proofs are equivalent for classical logic when both systems (inductive and circular) contain Peano arithmetic. This question is still open for linear logic: one essentially only knows that the provability of $\mu\mathsf{LL}_\infty$ circular proofs is strictly included in the provability of arbitrary non-wellfounded $\mu\mathsf{LL}_\infty$ proofs [20]. One inclusion is clear however: circular proof systems derives at least as many sequents as finitary one. Te see this, one proceeds by "unfolding" the (co)inductive inferences using the ability to build circular reasonings. In the case of $\mu\mathsf{MALL}$, a use of Park's rule [58] (rule (ν_{rec})):

$$\dfrac{\overset{\pi_1}{\vdash \Delta, A} \qquad \overset{\pi_2}{\vdash A^\perp, F\,[A/X]}}{\vdash \Delta, \nu X.F}\ (\nu_{\mathsf{rec}})$$

can be transformed into the derivation of Figure 1 (assuming $[\pi_1]$ and $[\pi_2]$ are obtained by recursively applying the same transformation). Little is known about the properties of such translations. The present paper aims at clarifying their properties at an operational level, to argue for their correctness from a Curry–Howard correspondence perspective. Hence we first need to develop a denotational semantics of $\mu\mathsf{LL}_\infty$.

$$
\dfrac{\begin{matrix}[\pi_1]\\ \vdash \Delta, A\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}[\pi_2]\\ \vdash A^{\perp}, F\,[A/X]\end{matrix} \qquad \dfrac{\vdash A^{\perp}, \nu X.F}{\vdash (F\,[A/X])^{\perp}, F\,[\nu X.F/X]}\,(\mathfrak{F}_F)}{\vdash A^{\perp}, F\,[\nu X.F/X]}\,(\mathsf{cut})}{\vdash A^{\perp}, \nu X.F}\,(\nu)}{\vdash \Delta, \nu X.F}\,(\mathsf{cut})
$$

Fig. 1: Circular unfolding of (ν_{rec})

c) Denotational semantics: While Tarskian semantics focus on the notion of truth and hence provability of formulas, denotational semantics examine the proof objects and the relationships between different proofs of a formula [46], [47], [49]. In particular, it is expected that the meaning of a proof remains unchanged during cut-reduction steps – a property commonly referred to as the *soundness theorem* of a denotational model.

The fact that linear logic can be viewed, from a proof-structural perspective, as a symmetrical refinement of intuitionistic logic is also reflected in the denotational models. Considering cartesian closed categories as models of intuitionistic logic [37]–[39], a proof in intuitionistic logic can be interpreted as a set function in the set-theoretical model. This interpretation is then generalized to use relations for interpreting proofs in linear logic [14], [15], [42]: the *relational model* will be an important and central example of our constructions.

An interesting question in denotational semantics is how to refine models to obtain syntactic properties in a syntax independent way. For instance, the notion of *totality* in denotational semantics offers a semantical account of the syntactic concept of proof normalization [30], [40], [48]. In this paper, we will address a notion of totality which allows us to distinguish pre-proofs from the valid ones.

d) Related works: Several previous works proposed denotational semantics for fixed-point logics. Santocanale and Fortier considered circular proofs for purely Additive linear logic with fixed-points and provided a categorical interpretation of circular proofs in μ-bicomplete categories [28], [28], [44]. In this paper we will consider full linear logic. Clairambault investigated [17], [18] the game with totality semantics of an extension of intuitionistic logic with fixed points in a finitary setting (independently of [29], [40]). Baelde *et al.* [4] provided a denotational semantics for $\mu\mathsf{MALL}$ (finitary) proofs in the setting of Girard's ludics [33]. Their interpretations of fixed points rules is based on ludics' designs which are infinitary objects, and the completeness result relies on finitization of infinitary objects. A categorical model of $\mu\mathsf{LL}$ is provided in [25] which is based on the standard notion of Seely category of classical linear logic and on strong functors acting on them.

e) Contributions: After recalling the necessary background in Section II, we first revisit the syntactical relationship between the finitary and circular proofs in Section III. The standard translation from $\mu\mathsf{MALL}$ to circular proofs is extended to $\mu\mathsf{LL}$. In the other direction, we present a translation from circular proofs to finitary ones under a condition called *strong validity*, extending results from Doumane's thesis [23].

In the same manner as we first define pre-proofs and then valid pre-proofs in syntax, we first provide a categorical semantics for non-wellfounded pre-proofs in Section IV, by considering a Cpo structure on the categorical model of $\mu\mathsf{LL}$ in [25]. We then refine our categorical model in Section V, based on the focused orthogonality construction given in [34], to capture the syntactic validity criterion. We exhibit two simple instances of our settings. The first one is based on the category $\mathbf{Rel}$ of sets and relations, and the second one is the category of $\mathbf{Nuts}$ (non-uniform totality spaces), an enrichment of $\mathbf{Rel}$ by considering sets equipped with an additional structure of *totality*. Finally, in Section VI, we consider the interpretation of the circular fragment: we relax our assumption of having a Cpo-enriched category, and we provide a parameterized interpretation of circular strongly valid pre-proofs, exploiting both validity and regularity in order to define inductively the interpretation function.

Our semantics is then used to investigate the denotational content of the standard translation from finitary proofs to non-wellfounded ones: we show that the above-mentioned translation from finitary proofs to circular ones preserves the semantics, suggesting that the translation is indeed correct (Theorem 16). In the other direction, we also show that the finitization translation of strongly valid pre-proofs also preserves the semantics (Theorem 26).

Moreover, the paper studies some properties of this semantics: the semantics is indeed sound in the sense that each element of an infinite cut-reduction sequence of pre-proofs converging to a cut-free valid pre-proof has the same interpretation as its limit (Corollary 15); we also show that valid pre-proofs are interpreted as morphisms in the focused orthogonality category. In the case of the concrete model $\mathbf{Nuts}$, although it is not true in general that the totality of the interpretation of a proof implies its validity, the notion of totality in $\mathbf{Nuts}$ provides the most liberal notion of validity as, intuitively, $\mathcal{T}(E)$ represents the total, that is, terminating computations of type E.

A pictorial summary of the results is shown in Figure 2. On the left side, we present the syntactical world and their relationship, and on the right side, the semantical world captures the different fragments of $\mu\mathsf{LL}_\infty$ and the relationship between circular strongly valid pre-proofs with $\mu\mathsf{LL}$ proofs.

A long version of this paper is available [26].

II. Background: $\mu\mathsf{LL}_\infty$ Proofs by Examples

In this section, we introduce the necessary background on finite, regular and non-wellfounded proof theory of linear logic with least and greatest fixed-points. We shall essentially work by illustrating the notions on examples rather than providing all the definitions, since all those notions are provided in the literature [6], [23]; we hope that this will make simpler to grasp those concepts. See long version [26] for details.

Linear logic (LL) was introduced by Jean-Yves Girard in his seminal work [31] soon after Kozen provided his axiomatization of the modal μ-calculus [36]. LL is a refinement of both

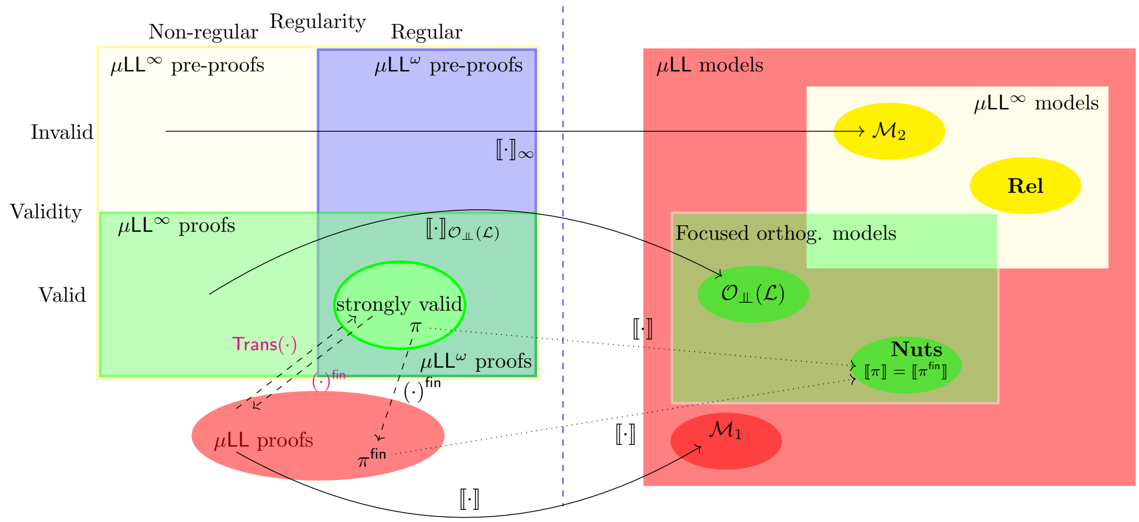


Fig. 2: Relationships between the proof systems and the semantics considered in the paper. Plain arrows refer to interpretation maps, dashed arrows refer to syntactic translations on proofs, dotted arrows refer to the interpretation of individual proofs.

classical and intuitionistic logic: LL is a substructural logic, meaning that one does not have free access to the structural rules of weakening and contraction. More precisely, we can only weaken and contract formulas if they have been marked with the so-called *exponential modalities*. The μ-calculus is a framework allowing to enrich a logic with expression of least and greatest fixed-points, that is speaking of inductive and coinductive properties. The remainder of this section recalls how one can extend LL with least and greatest fixed points operators, first introducing the syntax of formulas, then its finitary proof system and finally its infinitary proof system.

A. *Syntax of formulas of linear logic with fixed points*

Given a countable set of fixed-point variables $\mathcal{V}$ $(X, Y \dots)$, $\mu\mathsf{LL}$ ***pre-formulas*** are inductively defined as:

$$\begin{aligned} A, B, \dots := 1 \mid A \otimes B \mid 0 \mid A \oplus B \mid {!A} \mid \\ \bot \mid A \parr B \mid \top \mid A \,\&\, B \mid {?A} \mid \\ X \mid \mu X.A \mid \nu X.A. \end{aligned} \tag{1}$$

Natural numbers can be represented as $\mathsf{nat} = \mu X.(1 \oplus X)$.

Closed pre-formulas and capture-avoiding ***substitution***, $A[B/X]$, are defined as usual, μ and ν being the only two binders. We will almost exclusively use closed pre-formulas which are simply referred to as $\mu\mathsf{LL}$ ***formulas***. ***Negation*** (or *dualization*) is the *involution* extending LL negation[1] to pre-formulas extending with $(X)^\perp = X$, $(\mu X.A)^\perp = \nu X.(A)^\perp$. For instance $\mathsf{nat}^\perp = \nu X.(\bot \,\&\, X)$.

In addition to usual *sub-formulas*, we shall consider ***Fischer-Ladner subformulas***, a specific notion of sub-formula for the μ-calculus, for which the immediate FL-subformula of $\mu X.A$ is its unfolding, $A[\mu X.A/X]$. For instance $1 \oplus \mathsf{nat}$ is the immediate FL-subformula of nat. (Details are provided in the long version [26].)

[1] LL negation is the involution that swaps recursively the connectives of the second line of 1 with the one immediately above it: $(\bot)^\perp = 1, (A \parr B)^\perp = (A)^\perp \otimes (B)^\perp$...

Remark 1. *We will use in our proof of Theorem 20 an alternative notion of marked formula [23] (Definition 18) annotating the ν binder with an ordinal, as $\nu^\alpha X.F$.*

In the following sections, we shall consider two proof systems for deriving judgments concerning $\mu\mathsf{LL}$ formulas, a finitary proof system and a non-wellfounded one. Those proof systems derive ***sequents*** $\vdash \Gamma$ where Γ is an *ordered list* of $\mu\mathsf{LL}$ formulas. Using sequents as lists allows us to distinguish two different ***occurrences*** of the same formula in a sequent, by referring to their respective position in the sequent. The inference rules to be introduced in the following subsections will be equipped with a (standard [16]) notion of ***formula ancestor***, relating for each inference, occurrences of conclusion formulas to occurrences of formulas in the premisses. The ancestry relation is defined graphically in the proof system (as colored links) and will usually be kept implicit on examples unless useful. When a line links a context variable, say Γ, between a conclusion and a premise, this means that each formula of the list is in relation with the formula in the same position in the other list. While it is crucial to trace formula occurrences to give a computational and denotational content to proofs, in the following, it will also be exploited to define what is a *valid* non-wellfounded proof, using the notion of *threads*.

B. *Finitary $\mu\mathsf{LL}$*

In the present section, we will briefly describe the syntax of proofs of $\mu\mathsf{LL}$ [2], [35]. $\mu\mathsf{LL}$ proof system extends the usual one-sided sequent calculus of propositional LL [31], which is recalled in Figure 3(a), with the (μ) and (ν_{rec}) rules, given in Figure 3(b). As an example, consider the natural number formula $\mathsf{nat} = \mu X.(1 \oplus X)$, and its dual $\mathsf{nat}^\perp = \nu X.(\bot \,\&\, X)$.

(a)

$$\frac{}{\vdash F, F^{\perp}}\ (\mathsf{Ax}) \qquad \frac{\vdash \Gamma, F \quad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathsf{Cut}) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (\mathsf{X})$$

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_{\mathsf{i}}) \qquad \frac{}{\vdash \top, \Gamma}\ (\top) \qquad \text{(no rule for 0)}$$

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?\mathsf{d}) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!\mathsf{p}) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (?\mathsf{w}) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (?\mathsf{c})$$

(b)

$$\frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu) \qquad \frac{\vdash F[\nu X.F/X], \Gamma}{\vdash \nu X.F, \Gamma}\ (\nu) \qquad \frac{\vdash \Delta, A \quad \vdash ?\Gamma, A^{\perp}, F\,[A/X]}{\vdash \Delta, ?\Gamma, \nu X.F}\ (\nu_{\mathsf{rec}})$$

Fig. 3: (a) Inference rules of LL, (b) Inference rules for fixed-points

The following $\mu\mathsf{LL}$ proofs correspond to the encoding of the natural numbers and the successor function:

$$\pi_0 = \cfrac{\cfrac{\cfrac{}{\vdash 1}\ (1)}{\vdash 1 \oplus \mathsf{nat}}\ (\oplus_1)}{\vdash \mathsf{nat}}\ (\mu) \qquad \pi_{\mathsf{k}+1} = \cfrac{\cfrac{\pi_{\mathsf{k}}}{\vdash 1 \oplus \mathsf{nat}}\ (\oplus_2)}{\vdash \mathsf{nat}}\ (\mu)$$

$$\pi_{\mathsf{succ}} = \cfrac{\cfrac{\cfrac{}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\ (\mathsf{Ax})}{\vdash \mathsf{nat}^{\perp}, 1 \oplus \mathsf{nat}}\ (\oplus_2)}{\vdash \mathsf{nat}^{\perp}, \mathsf{nat}}\ (\mu)$$

C. *Non-wellfounded* LL *with fixed points* ($\mu\mathsf{LL}_\infty$)

Formulas and inferences of $\mu\mathsf{LL}_\infty$ are exactly the same as the one for $\mu\mathsf{LL}$ but except for the replacement of rule (ν_{rec}) by the simpler rule (ν), see Figure 3. The weakness of the inference for the greatest fixed-points is compensated by the fact that we consider derivations having infinite branches.

a) Pre-proofs and finite representations: A $\mu\mathsf{LL}_\infty$ ***pre-proof*** is a possibly infinite tree, generated by the inference rules of $\mu\mathsf{LL}_\infty$. Among all $\mu\mathsf{LL}_\infty$ pre-proofs, the ***regular*** (or ***circular***, or ***cyclic***) ones are those having finitely many distinct subtrees. Circular pre-proofs can be represented as finite proof-trees with back-edges or labels. Such a finite graph (*i.e.* a finite tree with back-edges) R can be unfolded to a unique $\mu\mathsf{LL}_\infty$ pre-proof that we call $\mathsf{Unfold}(R)$. This unfolding is of course non-injective and given a pre-proof π, any R such that $\mathsf{Unfold}(R) = \pi$ is called a ***finite representation*** of π. The necessary technical apparatus on those finite representations is given in Doumane's thesis [23]: we shall follow her definitions, recalling only the most important notions needed below .

But first, let us review examples of $\mu\mathsf{LL}_\infty$ derivations, regular or not. The proof in Figure 4a corresponds to the function on natural numbers which sends n to $(n \mod 3)$. On the other hand, non-wellfounded derivations can be unsound in general: for instance one can provide a pre-proof for any sequent $\vdash \Gamma$ (and in particular a pre-proof of the empty sequent $\vdash$) as in Figure 4b.

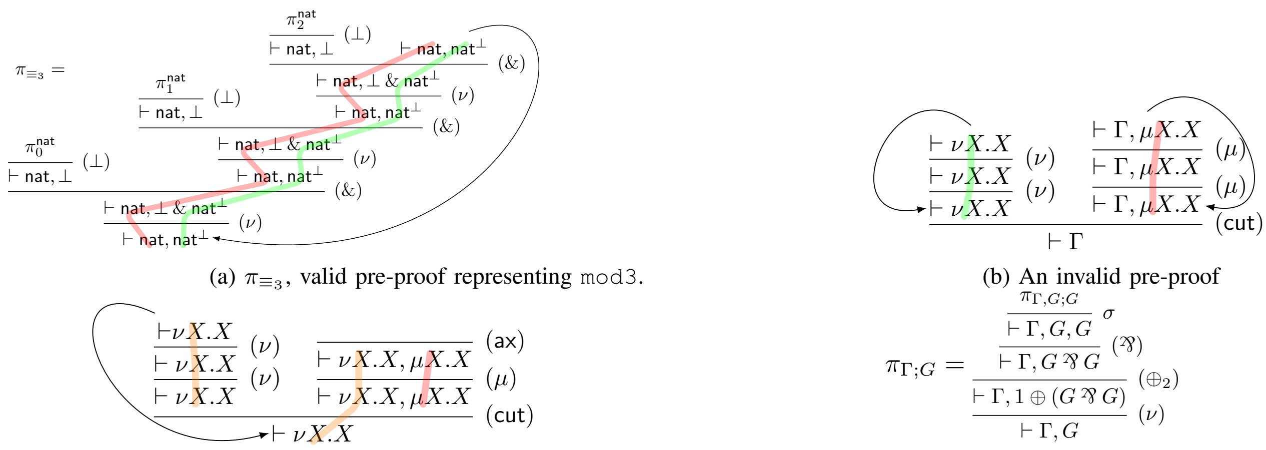
b) Thread, progress and validity: In [6], [23], a criterion, called ***validity (or progress) condition***, is introduced to distinguish proper proofs from pre-proofs. We review the criterion on some example, and refer to [6], [23], [43] for details.

A ***thread*** on an infinite branch $(\Gamma_i)_{\in\omega}$ of a pre-proof π is an infinite sequence of formula occurrences $t = (F_i)_{k\leq i\in\omega}$ such that for any $i \geq k$, $F_i \in \Gamma_i$ and F_{i+1} is an immediate ancestor of F_i. To each infinite branch one associates its set of threads: it may be empty, a singleton or a finite set (when equating threads having the same infinite suffix). For instance, the pre-proof in Figure 4a has only one infinite branch (the rightmost one) having exactly two threads starting from the root: $t_1 = \mathsf{nat}, \mathsf{nat}, \cdots$ (in red) and $t_2 = \mathsf{nat}^{\perp}, \bot \,\&\, \mathsf{nat}^{\perp}, \mathsf{nat}^{\perp}, \bot \,\&\, \mathsf{nat}^{\perp}, \cdots$ (in green) while the pre-proof in Figure 4c has an infinite branch containing no thread.

When there are only finitely many is such that F_i is principal in Γ_i, the thread is ***stationary***: this is the case of t_1 for instance (since nat is never principal in the branch). Similarly, the right branch of the pre-proof in Figure 4b has, from the right premise of the cut, one stationary thread for each formula in Γ together with one non-stationary thread $t_3 = \mu X.X, \mu X.X, \mu X.X, \cdots$. Given a thread t, we denote by $\mathsf{Inf}(t)$ the set of ***recurring formulas***, that occur infinitely often in t. For instance $\mathsf{Inf}(t_2) = \{\mathsf{nat}^{\perp}, \bot \& \mathsf{nat}^{\perp}\}$. We define ***valid*** (or ***progressing***) threads as those non-stationary threads t such that $\mathsf{Inf}(t)$ has as minimum (*wrt.* the sub-formula ordering) a ν-formula. *E.g.* t_2 is a valid thread since $\mathsf{nat}^{\perp}$ is a ν-formula and a subformula of $\bot \,\&\, \mathsf{nat}^{\perp}$, while t_1 and t_3 are not progressing as t_1 is stationary and $\mathsf{Inf}(t_3) = \{\mu X.X\}$. (We provide details on $\mu\mathsf{LL}_\infty$ subformulas and the minimality invoked above in the long version [26].)

A ***valid*** $\mu\mathsf{LL}_\infty$ ***proof*** (or $\mu\mathsf{LL}_\infty$ ***proof***, for short) is a pre-proof such that all its infinite branches contain a valid thread. *E.g.* the pre-proof in Figure 4a is a proof while that of Figure 4b is not. Indeed, the right-branch of the latter contains no valid thread: its only non-stationary thread, t_3, is not valid.

c) On validity of finite representations: The above notions of infinite branch, thread and validity on non-wellfounded proofs readily apply to the regular fragment and can naturally be adapted to ***finite representations*** of circular proofs. To emphasize that we refer to a finite representation, we shall in that case (as in [23] where the correspondence is formally established) speak of an ***infinite path***, a ***trace*** (ie. a sequence of ancestor-related formula occurrences of a finite



(a) $\pi_{\equiv_3}$, valid pre-proof representing `mod3`.

(b) An invalid pre-proof

(c) A pre-proof containing no threads

(d) A non regular pre-proof

Fig. 4: Some $\mu\mathsf{LL}_\infty$ regular derivations

representation) and of a ***valid trace*** (corresponding respectively to an infinite branch, a thread and a valid thread). Finite representations also allow us to consider some sets of branches having common properties, for instance by considering the *strongly connected component* of a finite representation.

d) Cut-reductions: The set of primitive (single step) reduction rules of $\mu\mathsf{LL}_\infty$ are the ones for LL plus the reduction in Figure 5 together with the corresponding commutation rules (See Figure 3.2 of [23]). Various cut-elimination theorems on non-wellfounded proofs are proved in [3], [6], [23] and especially of $\mu\mathsf{LL}_\infty$ itself [45] but the rest of the paper does not rely on those normalization results.

e) Functoriality: We end this section by stating the functoriality of $\mu\mathsf{LL}_\infty$ which we will use in Section III:

Proposition 1. *Let $(X, Y_1, \dots, Y_k)$ be a list of pairwise distinct propositional variables containing all the free variables of a formula F and let $\overrightarrow{C} = (C_1, \dots, C_k)$ be a sequence of closed formulas. The following rule is admissible in $\mu\mathsf{LL}_\infty$:*

$$\frac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (F[A/X, \overrightarrow{C}/\overrightarrow{Y}])^\perp, F[B/X, \overrightarrow{C}/\overrightarrow{Y}]}\ (\mathfrak{F}_F)$$

Proof. The proof is done by induction on the formula F (See Definition 2.38 of [23] for details). Exponentials do not modify the proof in any non trivial way [35]. $\square$

III. Circular vs Finitary proofs

Relating finitary proof and regular non-wellfounded proofs [9]–[12], [23], [51] is notoriously difficult. In this section, we will study the *syntactic relation* between the circular $\mu\mathsf{LL}_\infty$ proofs and $\mu\mathsf{LL}$ proofs by reviewing and extending known results about translations between finitary and circular proofs, in $\mu\mathsf{LL}$. Their *semantical relation* will be studied in Sections IV-B and VI-A.

A. Unfolding $\mu\mathsf{LL}$ proofs to circular proofs

As it is discussed in [23] for a wide class of fixed-point sequent calculi, provability of a sequent in a finitary sequent calculus with Park's rule entails its provability in the associated non-wellfounded sequent calculus. This can be done by translating a proof π of the finitary proof system to a circular proof, $\mathsf{Trans}(\pi)$, in the non-wellfounded proof system. Here, we straightforwardly adapt the result from Doumane's thesis to $\mu\mathsf{LL}$. (Our version of $\mu\mathsf{LL}$ differs from the $\mu\mathsf{LL}$ calculus considered in [23] as we use a more powerful (ν_{rec}) rule.)

Definition 2 ($\mathsf{Trans}(\pi)$)**.** *We define, by induction on the structure of a $\mu\mathsf{LL}$ proof π, a $\mu\mathsf{LL}_\infty$ pre-proof $\mathsf{Trans}(\pi)$ deriving the same sequent as π. We show only the (ν_{rec}) case as the other ones are trivially defined homomorphically:*

$$\mathsf{Trans}\left(\frac{\overset{\pi_1}{\vdash \Delta, A} \qquad \overset{\pi_2}{\vdash ?\Gamma, A^\perp, F[A/X]}}{\vdash \Delta, ?\Gamma, \nu X.F}\ (\nu_{\mathsf{rec}})\right) =$$

$$\dfrac{\overset{\mathsf{Trans}(\pi_1)}{\vdash \Delta, A} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathsf{Trans}(\pi_2)}{\vdash ?\Gamma, A^\perp, F[A/X]} \qquad \dfrac{\vdash ?\Gamma, A^\perp, \nu X.F}{\vdash ?\Gamma, (F[A/X])^\perp, F[\nu X.F/X]}\ (\mathfrak{F}_F)}{\vdash ?\Gamma, ?\Gamma, A^\perp, F[\nu X.F/X]}\ (\mathsf{cut})}{\vdash ?\Gamma, ?\Gamma, A^\perp, \nu X.F}\ (\nu)}{\vdash ?\Gamma, A^\perp, \nu X.F}\ (\mathsf{c})}{\vdash \Delta, ?\Gamma, \nu X.F}\ (\mathsf{cut})$$

The long version [26] provides details on this definition. The following is proved similarly to Proposition 2.14 of [23].

Proposition 3. *For any $\mu\mathsf{LL}$ proof $\pi \vdash \Gamma$, $\mathsf{Trans}(\pi)$ is a $\mu\mathsf{LL}_\infty$ proof of $\vdash \Gamma$.*

B. From circular to finitary proofs

Translating circular proofs to finitary ones is much more involved: it involves finding the appropriate co-inductive invariants for Park's rule (ν_{rec}) by looking at the circular proof, which does not contain such invariants.

While we do not know of any general translation from $\mu\mathsf{LL}_\infty$ circular proofs into finitary ones, there are however some proper fragments of $\mu\mathsf{LL}_\infty$ for which the structure of validity conditions is simple enough so as to allow us to extract invariants. The simplest such fragment corresponds to Santocanale and Fortier's setting [28] for circular proofs, that

$$
\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma, F\left[\mu X.F/X\right]}}{\vdash \Gamma, \mu X.F}\,(\mu) \qquad \dfrac{\overset{\pi'}{\vdash \Delta, F^{\perp}\left[\nu X.F^{\perp}/X\right]}}{\vdash \Delta, \nu X.F^{\perp}}\,(\nu)}{\vdash \Gamma, \Delta}\,(\mathsf{cut}) \quad \longmapsto_{\mathsf{cut}} \quad \dfrac{\overset{\pi}{\vdash \Gamma, F\left[\mu X.F/X\right]} \qquad \overset{\pi'}{\vdash \Delta, F^{\perp}\left[\nu X.F^{\perp}/X\right]}}{\vdash \Gamma, \Delta}\,(\mathsf{cut})
$$

Fig. 5: $(\mu) - (\nu)$ key cut-elimination reduction

is $\mu\mathsf{ALL}_\infty$ in which any circular proof can be finitized to a $\mu\mathsf{ALL}$ proof. This was partially extended to $\mu\mathsf{LL}_\infty$ as well by Doumane [23] considering a fragment of *translatable circular proofs* characterized by the condition that the proof has a circular representation in which along every infinite path, there exists a *strongly valid* trace t, *i.e.* such that exactly one formula of each recurring sequent of the *circular representation* is visited by t. Such a finitization was first considered in the study of an interpretation of $\mu\mathsf{MALL}$ in Ludics [5], and then used for the linear-time μ-calculus [24] and finally stated in a general way in Doumane's PhD [23] as the translatability condition. Below, we relax Doumane's condition and obtain a weaker but still sufficient condition to finitize $\mu\mathsf{LL}_\infty$ proofs: we finitize more circular $\mu\mathsf{LL}_\infty$ proofs.

Definition 4 (Trace-recurring formulas)**.** *Given a circular representation R, a sequent s in R and a trace t on R,* $\mathsf{Rec}(t, s)$ *as the set of formula occurrences of s which are visited infinitely often by t: an occurrence F of s belongs to* $\mathsf{Rec}(t, s)$ *if there are infinitely many i s.t. $t_i = (s, j)$, with F being the occurrence $s(j)$.*

Definition 5 (Strong validity)**.** *Let R be a circular representation and p an infinite path on R. A trace t is **strongly valid** if (i) t is valid and if for every sequent s of R which is conclusion of a ν-rule unfolding the minimal recurring formula of t,* $\mathsf{Rec}(t, s)$ *is a* singleton *and (ii) for each back-edge (s, s') in R,* $\mathsf{Rec}(t, s) = \mathsf{Rec}(t, s')$*. A finite representation R is **strongly valid**, if every infinite path in R admits a strongly valid trace. A circular pre-proof π is **strongly valid** if it admits a strongly valid finite representation.*

Example 6. *The proof in Figure 6a is strongly valid but not that in Figure 6b. The circular derivation in Figure 6b defines a valid proof π_∞ with $F = \mu X.((X \parr G) \& (X \parr H))$, $G = \nu X.(X \oplus \perp)$, $H = \nu X.(\perp \oplus X)$, $I = \mu Z.((Z \parr J) \oplus \perp)$, $J = \mu X.((K \parr X) \oplus \perp)$ and $K = \nu Y.\mu Z.((Z \parr \mu X(Y \parr X) \oplus \perp) \oplus \perp)$. π_∞ is an example of a valid circular proof having a complex validity structure, with three types of infinite branches, one for each strongly connected component [43]. (See the long version [26] for more details.)*

In particular, paths visiting infinitely all sequents of the finite representation *are validated by a trace that progresses on K but that visits both I, J, and K; we mark in red in Figure 6b where the strong validity condition is violated.*

On the other hand, that the proof of Figure 6a is not strongly valid for Doumane's definition of translatability *as the only trace validating a path visiting infinitely all sequents has V as minimal formula and must visit both occurrences of V in sequent $\vdash B, V, V$, which is forbidden in the criterion from [23]. (See the long version [26] for more in-depth discussions of the difference between the two criteria.)*

Strong validity is a sufficient condition for finitization:

Proposition 7. *If $\vdash \Gamma$ has a strongly valid proof in $\mu\mathsf{LL}_\infty$, it is provable in $\mu\mathsf{LL}$.*

In the following, we will show how to build a $\mu\mathsf{LL}$ proof π^{fin} given any $\mu\mathsf{LL}_\infty$ strongly valid proof. We first recall some results by Doumane that apply independently of the extension of the criterion. We then explain our finitization process which generalizes Doumane's translatability criterion. Note that, while our construction extends very significantly her previous results [23], our proof follows the exact same ideas and does not present much difficulties.

As a consequence, we will mostly focus below on the definition of $\Uparrow(R)$ that will be used in the following sections investigating the semantics of proofs.

Definition 8 (Invariant formula)**.** *Let $\nu X.F, \Gamma$ be a list of $\mu\mathsf{LL}$ formulas. The **invariant formula** I^F_Γ is $\nu X.(F \oplus (\parr\Gamma)^\perp)$.*

Proposition 9 ([23], Prop. 2.15)**.** *The following rules are $\mu\mathsf{LL}$-derivable:*

$$
\dfrac{\vdash \Delta\left[\nu X.F/X\right]}{\vdash \Delta\left[I^F_\Gamma/X\right]}\,(\mathsf{subst}) \qquad \dfrac{}{\vdash I^F_\Gamma, \Gamma}\,(\mathsf{close}) \qquad \dfrac{\vdash \Delta, F\left[I^F_\Gamma/X\right]}{\vdash \Delta, I^F_\Gamma}\,(\mathsf{unfold})
$$

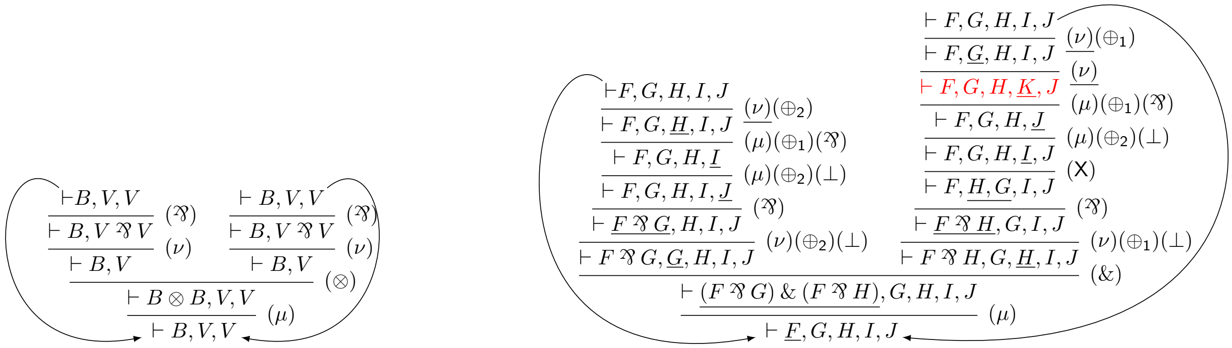
Moreover, (subst) *and* (unfold) *are **circularly derivable** in $\mu\mathsf{LL}_\infty$.*

In order to show the finitization results, we adopt the same measures on finite representations $\mathsf{size}(R)$ as it is done Definition 2.45 of [23]:

Definition 10. *Let* $\mathsf{nax}(\pi)$ *and* $\mathsf{elc}(\pi)$ *be the numbers of the non-axiom rules in π and the numbers of the elementary cycles in π respectively.* $\mathsf{size}(\pi)$ *is defined as the pair* $(\mathsf{elc}(\pi), \mathsf{nax}(\pi))$*, ordered lexicographically.*

The finitization process "propagates" the invariant formula in the circular proof so as to *remove back-edges* using (close). We now define the proof pattern, $\Uparrow(R)$, which performs this task and will serve to interpret circular proofs:

Definition 11. *Let P be a strongly connected finite representation of a strongly valid proof of $\vdash \Gamma, \nu X.F$ and t be a strongly valid trace of minimal recurring formula $\nu X.F$ visiting every sequent of P infinitely often. Let P be as follows: We define finite representation $\Uparrow(R)$ of conclusion $\vdash \Gamma, F\left[I^F_\Gamma/X\right]$ (for R the premise of the concluding (ν) of P) inductively on the structure of R (disregarding the back-edges from the inductive tree structure of course) and by case on the last rule. We shall maintain the following invariant : if S has conclusion sequent $s = \vdash \Delta, \Sigma\left[\nu X.F/X\right]$ where* $\mathsf{Rec}(t, s) = \Sigma\left[\nu X.F/X\right]$ *(Σ may*



(a) A strongly valid proof.

(b) The valid but not strongly valid proof π_∞.

Fig. 6

be empty), then $\Uparrow(S)$ has conclusion $\vdash \Delta, \Sigma\left[I_\Gamma^F/X\right]$. $\Uparrow(S)$ is defined as:

- *Base case: that is if s is source of a back-edge. If the back-edge points to the root i.e. $\Delta = \Gamma, \Sigma = X$ then $\Uparrow(S)$ is the derivation of the* (close) *given by Proposition 9. Otherwise, it points to some other node, in which case we let the back-edge as is (indeed, by the invariant we maintain, the target of the back-edge is $\vdash \Delta, \Sigma\left[I_\Gamma^F/X\right]$).*
- *If $S = \dfrac{\left(\dfrac{R'_l}{\vdash \Delta_l, \Sigma_l\left[\nu X.F/X\right], \Xi_l\left[\nu X.F/X\right]}\right)_{l\in L}}{\vdash \Delta, \Sigma\left[\nu X.F/X\right]}\,(r)$ with $\mathsf{Rec}(t,s) = \Sigma\left[\nu X.F/X\right]$ and $\mathsf{Rec}(t, \vdash \Delta_l, \Sigma_l\left[\nu X.F/X\right], \Xi_l\left[\nu X.F/X\right]) = \Sigma_l\left[\nu X.F/X\right]$:*
  - *(i) if $r = (\nu)$ unfolding $\nu X.F$ (then $L = \{\star\}$, $\Sigma_\star = \Sigma = X$ and $\Xi_\star = \emptyset$) $\Uparrow(S) = \dfrac{\dfrac{\Uparrow(R')}{\vdash \Gamma, F\left[I_\Gamma^F/X\right]}}{\vdash \Gamma, I_\Gamma^F}\,(\mathsf{unfold})$*

    otherwise
  - *(ii) $\Uparrow(S) = \dfrac{\left(\dfrac{\dfrac{\Uparrow(R'_l)}{\vdash \Delta_l, \Sigma_l\left[I_\Gamma^F/X\right], \Xi_l\left[\nu X.F/X\right]}}{\vdash \Delta_l, \Sigma_l\left[I_\Gamma^F/X\right], \Xi_l\left[I_\Gamma^F/X\right]}\,(\mathsf{subst})\right)_{l\in L}}{\vdash \Delta, \Sigma\left[I_\Gamma^F/X\right]}\,(r)$*

Thanks to the previous definition, we can now easily prove Proposition 7. (See details in the long version [26].)

Proof of Proposition 7. The proof goes by induction on $\mathsf{size}(\pi)$ with a base case when $\mathsf{elc}(\pi) = 0$: in that case, π has no back-edge as the finitization is the identity map. Otherwise, there are two cases: either the finite representation, R, associated to π is strongly connected as graph or it is not.

$\triangleright$ Assuming that R is strongly connected. Then, there is an infinite path p that visits all the sequents of R and an associated strongly valid trace t of minimal formula $\nu X A$. Wlog, assume that a sequent where the minimal formula of t has been unfolded, $\vdash \Gamma^\perp, \nu X A$, is the conclusion of R. We are in the situation of Definition 11.

We can now consider the strongly valid finite representation $\Uparrow(R)$ of $\vdash \Gamma^\perp, A\left[I_\Gamma^A/X\right]$. The complexity of the proof $\Uparrow(\pi)$ is strictly less than that of π, since $\mathsf{elc}(\Uparrow(\pi)) < \mathsf{elc}(\pi)$. So, by induction hypothesis, there is a μLL (finite) proof ρ of $\vdash \Gamma^\perp, A\left[I_\Gamma^A/X\right]$. In this case, the π^{fin} is defined as follows where the rightmost proof of $\vdash I_\Gamma^A, \Gamma^\perp$ is the derived rule (close) from Proposition 9 and A_I denotes $A\left[I_\Gamma^A/X\right]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A_I^\perp, A_I}\,(\mathsf{ax}) \qquad \dfrac{\rho}{\vdash \Gamma^\perp, A_I}}{\vdash A^\perp\left[(I_\Gamma^A)^\perp/X\right] \,\&\, \Gamma^\perp, A_I}\,(\&)}{\vdash (I_\Gamma^A)^\perp, A\left[I_\Gamma^A/X\right]}\,(\mu) \qquad \dfrac{\dfrac{\dfrac{}{\vdash \Gamma, \Gamma^\perp}\,(\mathsf{ax})}{\vdash A\left[I_\Gamma^A/X\right] \oplus \Gamma, \Gamma^\perp}\,(\oplus_2)}{\vdash I_\Gamma^A, \Gamma^\perp}\,(\nu)}{\vdash \Gamma^\perp, \nu X A}\,(\nu_{\mathsf{rec}}) \tag{2}$$

$\triangleright$ We now consider the case that R is not strongly connected, then there are two sequents $\vdash \Gamma$ and $\vdash \Delta$ such that there is no path from $\vdash \Gamma$ to $\vdash \Delta$. Let R_1 be the part of R which is reachable from $\vdash \Gamma$, and let R_2 be obtained from R by adding an auxiliary rule r on $\vdash \Gamma$ and taking the reachable part from the conclusion of R. R_1, R_2 respectively correspond to strongly valid circular proofs π_1 and π_2. Since R_1 does not have $\vdash \Gamma$ among its non-axiomatic rules, we have $\mathsf{nax}(R_1) < \mathsf{nax}(R)$, and then by induction hypothesis we have π_1^{fin} a finitization of π_1. By removing $\vdash \Gamma$ from R_2, we have $\mathsf{nax}(R_2) < \mathsf{nax}(R)$. Hence, by induction hypothesis, we have π_2^{fin}, a finitization of π_1. As π^{fin} is simply defined by plugging two proofs π_1^{fin} and π_2^{fin} at the assumption leaf introduced above. ☐

Remark 2. *To keep the presentation simple, we made a slight simplification on finite representations of circular proofs referring to finite trees with back-edges which are simpler to describe, more intuitive and sufficient to explain our results. Still, the actual representation used for circular representations is that of a graph with input node, it is for instance needed to carry out the induction of the previous proof. For instance, any circular proof can be represented in the graph-formalism of such that it corresponds to a finite tree with back-edges. (meaning that for all nodes, all incoming edges but one are "renaming rules" used in Def 2.30 of [23].)*

Remark 3. *From Definition 11, we can define π^{fin} from a strongly valid circular representation of π by induction on* $\mathsf{size}(\pi)$. *Notice that the finite proof π^{fin} is not uniquely defined for a given strongly valid proof: it depends (i) on a choice of a finite representation of π, (ii) on a set of strongly valid traces* *and (iii) on a choice of* one *strongly valid trace for each strongly connected component of R.*

Notice also that our class of strongly valid proofs obviously contains all the unfoldings of finitary proofs (already included in Doumane's translatable proofs). Notice that the above method does not apply to π_∞ (defined in Figure 6b).

IV. Semantics of $\mu\mathsf{LL}_\infty$ pre-proofs

In this section, we show that one can obtain a categorical axiomatization of models of $\mu\mathsf{LL}_\infty$ pre-proofs by assuming a Cpo structure on the categorical model of $\mu\mathsf{LL}$ [25].

Definition 12. *A categorical model of $\mu\mathsf{LL}_\infty$ pre-proofs is a pair $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ where*

1) *$\mathcal{L}$ is a model of linear logic, i.e. a Seely category [42].*
2) *$\overrightarrow{\mathcal{L}} = (\mathcal{L}_n)_{n\in\mathbb{N}}$ with $\mathcal{L}_n$ a class of strong functors $\mathcal{L}^n \to \mathcal{L}$, and $\mathcal{L}_0 = \mathsf{Obj}(\mathcal{L})$.*
3) *all k projection strong functors $\mathcal{L}^k \to \mathcal{L}$ belong to $\mathcal{L}_k$ as well as $\mathbb{X} \circ \overrightarrow{\mathbb{X}}$ for $\mathbb{X} \in \mathcal{L}_n$ and $\mathbb{X}_i \in \mathcal{L}_k$ for $1 \leq i \leq n$.*
4) *the strong functors $\otimes$ and $\&$ belong to $\mathcal{L}_2$, the strong functor $!\_$ belongs to $\mathcal{L}_1$ and, if $\mathbb{X} \in \mathcal{L}_n$, then $(\mathbb{X})^\perp \in \mathcal{L}_n$.*
5) *for all $\mathbb{X} \in \mathcal{L}_1$ the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{X}})$ of coalgebras of the functor $\overline{\mathbb{X}}$ [2] has a final object. Moreover, for any $\mathbb{X} \in \mathcal{L}_{k+1}$, the associated strong functor $\nu\mathbb{X} : \mathcal{L}^k \to \mathcal{L}$ belongs to $\mathcal{L}_k$.*
6) *and last, $\mathcal{L}$ is a Cpo-enriched category s.t. greatest lower bounds of any non-empty set exist in each hom-set*

Items 1-5 define what is known as a model of $\mu\mathsf{LL}$ [25].

We interpret a formula A with repetition-free sequence $\overrightarrow{X} = (X_1, \ldots, X_k)$ of fixed-point variables containing all free variables of A as an element in $\mathcal{L}^k$, called $[\![A]\!]_{\overrightarrow{X}}$ by induction on the formulas in the obvious way, *e.g.* $[\![A \otimes B]\!]_{\overrightarrow{X}} = \otimes \circ ([\![A]\!]_{\overrightarrow{X}}, [\![B]\!]_{\overrightarrow{X}})$ considering $\otimes \in \mathcal{L}_2$, and $[\![\nu X.A]\!]_{\overrightarrow{X}} = \nu([\![A]\!]_{\overrightarrow{X},X})$ [3]. Then one also has $[\![A^\perp]\!]_{\overrightarrow{X}} = ([\![A]\!]_{\overrightarrow{X}})^\perp$ up to a natural isomorphism which allows us to define other formulas by De Morgan duality.

The interpretation a $\mu\mathsf{LL}_\infty$ pre-proof π follows the classical approach of understanding an infinite object as the limit of its finite approximations. As usual to handle this approximation we assume to have this rule: $\dfrac{}{\Gamma}\ (\Omega)$ for any sequent Γ in the inference rules of $\mu\mathsf{LL}_\infty$, and we interpret (Ω) as the least element of $\mathcal{L}(1,1)$. The interpretation of our finite approximations requires semantics of LL rules which is given, for instance in [42], and we also recall them in the long version, plus interpretation of the (ν) and (μ) rules:

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F\,[\mu X.F/X]}}{\vdash \Gamma, \mu X.F}\ (\mu)\right]\!\!\right] = [\![\pi]\!] \qquad \left[\!\!\left[\dfrac{\dfrac{\pi}{\vdash \Gamma, F\,[\nu X.F/X]}}{\vdash \Gamma, \nu X.F}\ (\nu)\right]\!\!\right] = [\![\pi]\!]$$

[2] $\overline{\mathbb{X}}$ is the the underlying functor of the strong functor $\mathbb{X}$.

[3] We assume that the iso between νF and $F(\nu F)$ is always the identity as this holds in our concrete models. This assumption is highly debatable from the view point of category theory where the notion of equality of objects is not really meaningful. It will be dropped in a longer version of this paper.

Definition 13. *Let π be a $\mu\mathsf{LL}_\infty$ pre-proof of $\vdash \Gamma$, we define $[\![\pi]\!]_\infty$ as $\bigcup_{\rho\in\mathsf{fin}(\pi)}[\![\rho]\!]$ where $\mathsf{fin}(\pi)$ is the set of all finite sub-pre-proofs of π (we can do this, thanks to the (Ω) rule), and $\bigcup$ is the supremum of the directed subsets in $\mathcal{L}(1, [\![\Gamma]\!])$.*

From now on, we drop the subscript ∞ (as for other models later on) from notation $[\![\_]\!]$ whenever it is clear from context.

The soundness of our semantics follows a general result on the semantics of any Cauchy sequence of pre-proofs (Lemma 14): the set of $\mu\mathsf{LL}_\infty$ pre-proofs is the metric completion of finite proofs, a standard in the literature [8], [54], [56], with respect to the metric on pre-proofs given in Section II.

Lemma 14. *Let (π_i) be a Cauchy sequence. Then $[\![\lim_{n\to\infty}\pi_i]\!] = \bigcup_i \bigcap_{j>i}[\![\pi_j]\!]$.*

Proof sketch. The proof amounts to noting that for each finite approximant π' of the limit proof $\lim_{n\to\infty}\pi_i$, there exists an i such that for all $j > i$, π' is a finite sub-pre-proof of all π_j. Hence $[\![\pi']\!]$ is less than $[\![\pi_j]\!]$ for all $j > i$, so, $[\![\pi']\!] \subseteq \bigcap_{j>i}[\![\pi_j]\!]$ (here $\subseteq$ denotes the partial order on the hom-sets). $\square$

Corollary 15. *If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of $\mu\mathsf{LL}_\infty$, then $[\![\pi]\!] = [\![\pi']\!]$.*

A. $\mathbf{Rel}$ as a concrete model of $\mu\mathsf{LL}_\infty$

Let $\mathbf{Rel}_n$ be the class of all n-ary strong functors F where $\overline{F}$ is a locally continuous and strict in the sense that it maps inclusions to inclusions, and for all $\overrightarrow{E}, \overrightarrow{F} \in \mathbf{Rel}^n$ and all directed set $D \subseteq \mathbf{Rel}^n(\overrightarrow{E}, \overrightarrow{F})$, one has $\mathbb{F}(\bigcup D) = \bigcup\{\mathbb{F}(\overrightarrow{s}) \mid \overrightarrow{s} \in D\}$. We know that $(\mathbf{Rel}, (\mathbf{Rel}_n)_{n\in\mathbb{N}})$ is a model of $\mu\mathsf{LL}$ [25]. Since $\mathbf{Rel}$ is a Cpo enriched category such that any non-empty subset of each hom-set has glb, $(\mathbf{Rel}, (\mathbf{Rel}_n)_{n\in\mathbb{N}})$ is also a model of $\mu\mathsf{LL}_\infty$.

As an example, consider the circular proof $\pi_{\equiv_3}$ (Figure 4a). The interpretation of π_k^{nat} in $\mathbf{Rel}$ is, up to an iso, the natural number k, and we denote it by $\underline{k}$, i.e $[\![\pi_k^{\mathsf{nat}}]\!]_{\mathbf{Rel}} = \underline{k}$. To compute interpretation of $\pi_{\equiv_3}$, we need to take supremum of the interpretation of all finite sub pre-proofs. For example, imagine that in the proof $\pi_{\equiv_3}$ above, we do a Ω rule instead of the back-edge, and called this proof σ. Then we have $[\![\sigma]\!]_{\mathbf{Rel}} = \{(\underline{2}, (2,(2,(1,*))), (\underline{1}, (2,(1,*))), (\underline{0}, (1,*)))\}$, so, up to an iso we have $[\![\sigma]\!]_{\mathbf{Rel}} = \{(\underline{2}, \underline{2}), (\underline{1}, \underline{1}), (\underline{0}, \underline{0})\}$. If we do one more step, then $(\underline{0}, \underline{3}) \in [\![\pi_{\equiv_3}]\!]_{\mathbf{Rel}}$. Hence, $[\![\pi_{\equiv_3}]\!]_{\mathbf{Rel}} = \{(\underline{n}, \underline{m}) \mid \underline{n} = \underline{m} \mod 3\}$.

Another example of a $\mu\mathsf{LL}_\infty$ model is coherence spaces that we do not discuss further in the present paper.

B. Preservation of the interpretation by unfolding finite proofs

We can now prove that our semantics is preserved via the operation $\mathsf{Trans}\,()$ (see Section III). Notice that if we associate a system of equations on the morphisms of the category $\mathcal{L}$ to a circular proof, then the interpretation from Definition 13 is a solution of the corresponding system of equations.

Theorem 16. *Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model, π a $\mu\mathsf{LL}$ proof. Then $[\![\pi]\!] = [\![\mathsf{Trans}\,(\pi)]\!]_\infty$ where $[\![\_]\!]$ is the semantics in a $\mu\mathsf{LL}$ model.*

Proof sketch. The proof goes by induction on the structure of π and case distinction on its last rule, using functoriality and using the universal property of the final coalgebra. Details of the proof are provided in the long version. $\square$

V. Semantics of $\mu\mathsf{LL}_\infty$ proofs

In the previous section, we provided the interpretation of pre-proofs, disregarding proof validity. In this section, we will provide a refinement of our $\mu\mathsf{LL}_\infty$ model based on the orthogonality construction given in [34], and show that valid proofs will be interpreted as morphisms in the orthogonality category where the orthogonality relation satisfies a property called ***focused orthogonality***. Nevertheless the model of pre-proofs given in Section IV is an important step, as the interpretation of proofs in the more-refined focused orthogonality model will be the same as their interpretation, considered as pre-proofs, in the CPO model. Hence, results of the CPO model such as soundness (Corollary 15) and preservation of semantics can be applied directly (Theorem 16) to this orthogonality model.

A. Preliminaries on orthogonality categories

We first recall some definitions (see [34] for more details). Let $\mathcal{L}$ be a *-autonomous category with monoidal units 1 and $\perp$. An ***orthogonality relation*** is a family of subsets $\perp_c \subseteq \mathcal{L}(1,c) \times \mathcal{L}(c,\perp)$ indexed by objects $c \in \mathcal{L}$ and satisfying some compatibility conditions with respect to the linear logic structure [34]. For a subset $X \subseteq \mathcal{L}(1,c)$, its orthogonal $X^\perp$ is $X^\perp := \{y : c \to \perp \mid \forall x \in X\,(x \perp_c y)\}$. And dually, for a subset $Y \subseteq \mathcal{L}(c,\perp)$, we have $Y^\perp := \{x : 1 \to c \mid \forall y \in Y\,(x \perp_c y)\}$. Finally we denote by $\mathcal{D}(c)$ the set $\{X \subseteq \mathcal{L}(1,c) \mid X = X^{\perp\perp}\}$: one can see that $\mathcal{D}(c)$ is a complete lattice. In this paper, we will restrict to the special case where the orthogonality relation arises from a distinguished subset $\perp\!\!\!\perp \subset \mathcal{L}(1,\perp)$, the ***pole***, as follows: $\perp_c := \{(x,y) \in \mathcal{L}(1,c) \times \mathcal{L}(c,\perp) \mid y \circ x \in \perp\!\!\!\perp\}$.

We then define the ***focused orthogonality category*** [34] as:

Definition 17. *The focused orthogonality category $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ of a category $\mathcal{L}$ with $\perp\!\!\!\perp \subset \mathcal{L}(1,\perp)$ has as objects the pairs (c,X) with $c \in \mathcal{L}$, $X \in \mathcal{D}(c)$, and as morphisms $f : (c,X) \to (d,Y)$ the $f : c \to d$ in $\mathcal{L}$ s.t. $\forall x \in X.\ f \circ x \in Y$.*

B. Semantics of $\mu\mathsf{LL}_\infty$ in $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$

1) Interpretation of formulas: Given a closed $\mu\mathsf{LL}_\infty$ formula A, we denote by $[\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ the interpretation of A in $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$, so that $[\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ is a pair $([\![A]\!]_{\mathcal{L}}, \mathcal{O}([\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}))$ where $\mathcal{O}([\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}) \in \mathcal{D}([\![A]\!]_{\mathcal{L}})$.

Let $(\mathcal{L}, \vec{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model with a pole $\perp\!\!\!\perp \subseteq \mathcal{L}(1,\perp)$. We know how to interpret the LL formulas in $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ using Theorem 54 of [34]. For the fixed-point formulas $\mu X A$ and $\nu X A$, we know, by induction hypothesis, that $[\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ exists and it is lifting of the functor $[\![A]\!]_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ in the following sense where U is the forgetful functor, as depicted on the right. Using Corollary 3.4 of [27], we know that the initial algebra and final coalgebra of the endofunctor $[\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ exist,

$$\begin{tikzcd} \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L}) \arrow[r, "{[\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}}"] \arrow[d, "U"'] & \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L}) \arrow[d, "U"] \\ \mathcal{L} \arrow[r, "{[\![A]\!]_{\mathcal{L}}}"'] & \mathcal{L} \end{tikzcd}$$

and we take them respectively as the interpretation of $\mu X A$ and $\nu X A$ [4]. (We included the construction of $[\![\mu X A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ and $[\![\mu X A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ in the long version, as they will be used in the proof of Lemma 19.)

2) Interpretation of proofs: In section IV, we defined the interpretation of a pre-proof π of $\vdash \Gamma$ as a morphism $[\![\pi]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ in $\mathcal{L}(1, [\![\Gamma]\!])$. We will prove that if the proof π is a valid proof, then $[\![\pi]\!]$ is a morphism in the orthogonality category $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$. We simplify the notation $[\![A]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ by $[\![A]\!]$

The proof method is similar to the proof of soundness of LKID^ω in [11]. However the system of [11] is classical logic with inductive definitions, and their proof is for a Tarskian semantics. We need to adapt that proof in two aspects: considering $\mu\mathsf{LL}_\infty$ instead of LKID^ω, and trying to deal with a denotational semantics instead of a Tarskian semantics. The adaptation for $\mu\mathsf{LL}_\infty$ is somehow done in [23], as there is soundness theorem for $\mu\mathsf{MALL}_\infty$ with respect to the truncated truth semantics (a Tarskian semantics). So, basically, the main point of our proof is turning a Tarskian soundness theorem into a denotational soundness theorem. Following this approach, the crucial lemma is Lemma 19 which its proofs is essentially constructing an infinite branch inductively using properties of orthogonality, and also providing a non-increasing sequence of ordinals that decreases infinitely often so that it ends the proof by contradiction.

We first borrowed the following definition from [23].

Definition 18. *The **marked formulas** of $\mu\mathsf{LL}_\infty$ are defined as:*

$$\begin{aligned} A, B, \cdots := \ & 1 \mid 0 \mid \perp \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \\ & \mid A \parr B \mid ?A \mid !B \mid X \mid \mu X.F \mid \nu^\alpha X.F \end{aligned} \tag{3}$$

for α an ordinal. A° denotes the label-stripped formula A.

In $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$, $\nu^\alpha X.F$ is interpreted as $([\![\nu X.F^\circ]\!]_{\mathcal{L}}, U_\alpha)$; other marked formulas are interpreted as usual.

To complete the proof of our goal, we need the following crucial lemma which its proof follows essentially constructing an infinite branch inductively using properties of the orthogonality, and providing a non-increasing sequence of ordinals that decreases infinitely often to end the proof by contradiction. (Full details are provided in the long version.)

Lemma 19. *If π is a proof of $\vdash \Gamma$ and $[\![\pi]\!] \notin \mathcal{O}(([\![\Gamma]\!]))$, then*

1) *π has an infinite branch $\gamma = (\vdash \Gamma_i)_{i\in\omega}$ such that $[\![\pi_i]\!] \notin \mathcal{O}([\![\Gamma_i]\!])$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$*
2) *and there exists a sequence of functions $(f_i)_{i\in\omega}$ where f_i maps formulas D of Γ_i to a marked formula $f_i(D)$ s.t. (i) $(f_i(D))^\circ = D$, (ii) one can write $\Gamma_i = \Gamma'_i, C$, (iii) and there exists $x \in \mathcal{O}([\![(f_i(\Gamma'_i))^\perp]\!])$ such that $[\![\pi_i]\!].x \notin \mathcal{O}([\![f_i(C)]\!])$ where $\Gamma'_i = A^i_1, \cdots, A^i_{n_i}$ and $[\![(f_i(\Gamma'_i))^\perp]\!] = ([\![f_i(A^i_1)]\!])^\perp \otimes \cdots \otimes ([\![f_i(A^i_{n_i})]\!])^\perp$.*

Now, we can state and prove our main result of this section.

Theorem 20. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{O}([\![\Gamma]\!])$.*

[4]To have a more simple notation, we have only provided the interpretation of formulas with a single free variable. One can do it for any formulas in the obvious way.

Proof. Let us assume $[\![\pi]\!] \notin \mathcal{O}([\![\Gamma]\!])$. We can then apply Lemma 19 to obtain an infinite branch $(\vdash \Gamma_i)_{i\in\omega}$ and a sequence $(f_i)_{i\in\omega}$ satisfying properties 1 and 2 of Lemma 19. By the definition of a valid proof, there exists a valid thread $t = (F_i)_{i\in\omega}$ for the infinite branch $(\vdash \Gamma_i)_{i\in\omega}$. Let νXF be the minimal formula of t. So, there are infinitely many times in t that we use a ν rule to unfold νXF. Let $(i_k)_{k\in\omega}$ be the sequence of indices where νXF gets unfolded. Then νXF in the sequent Γ_{i_k} is a sub-occurrence of νXF in the sequent $\Gamma_{i_{k'}}$ for $k \geqslant k'$. By the property 2 of Lemma 19, $f_{i_k}(\nu XF) = \nu^{\alpha_k} X.f_{i_k}(F)$. Therefore, by the property 2 of Lemma 19 and by the construction of the f_i in the proof of Lemma 19, the sequence $(\alpha_k)_{k\in\omega}$ is strictly decreasing. This contradicts the well-foundedness property of the ordinals: we conclude that $[\![\pi]\!] \in \mathcal{O}([\![\Gamma]\!])$. □

We denote by $1_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$ the unit $(1, \mathcal{O}(1))$ of the tensor in the category $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$.

Corollary 21. *If π is a valid proof of $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})(1_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}, [\![\Gamma]\!])$.*

Proof. Let us assume $[\![\pi]\!] \notin \mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})(1_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}, [\![\Gamma]\!])$. So, there is $x \in \mathcal{O}(1)$ such that $[\![\pi]\!] \circ x \notin \mathcal{O}([\![\Gamma]\!])$. We know that $\mathcal{D}(1) = \{X \subseteq \mathcal{L}(1,1) \mid X = X^{\perp\perp}\} = \{\{\mathsf{Id}_1\}\}$. So, $x = \mathsf{Id}_1$, and $[\![\pi]\!] \circ x = [\![\pi]\!] \notin \mathcal{O}(\Gamma)$ which contradicts Theorem 20. □

Remark 4. *The fact that we have considered focused orthogonality is important in our work, as we use it a lot in the proof of Lemma 19. This assumption is also essential in the construction of fixed-point in [27].*

Remark 5. *The category $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ is not necessarily a $\mu\mathsf{LL}_\infty$ model in the sense of Definition 12, as it can be a non cpo-enriched category. We will see an example of this in Section V-C. Nonetheless, the interpretation of $\mu\mathsf{LL}_\infty$ proofs are the same in both categories, i.e. $[\![\pi]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})} = [\![\pi]\!]_{\mathcal{L}}$.*

We end this section by a corollary about the notion of *bouncing validity*, which generalize the validity criterion in order to take into account axioms and cuts rules [3].

Corollary 22. *Let π be a bouncing valid proof of the sequent $\vdash \Gamma$. $[\![\pi]\!] \in \mathcal{O}([\![\Gamma]\!])$.*

Proof. By the cut-elimination theorem of bouncing proofs (Theorem 5.1 of [3]), π will be reduce to a valid cut-free π'. By Corollary 15, we know that $[\![\pi]\!] = [\![\pi']\!]$. Moreover, we have $[\![\pi']\!] \in \mathcal{O}([\![\Gamma]\!])$ by Theorem 20. Hence $[\![\pi]\!] \in \mathcal{O}([\![\Gamma]\!])$. □

C. Valid proofs as total elements

If we consider the category $\mathbf{Rel}$ and the pole $\perp\!\!\!\perp_{\mathbf{Rel}} = \{\{\mathsf{id}\}\}$, the category $\mathcal{O}_{\perp\!\!\!\perp_{\mathbf{Rel}}}(\mathbf{Rel})$ is the category of non-uniform totality spaces ($\mathbf{Nuts}$) studied in [25]. Explicitly, for a set A and a subset $X \subseteq \mathbf{Rel}(1, A) = \mathcal{P}(A)$, one has $X^\perp = \{u' \subseteq A \mid \forall u \in X \; u \cap u' \neq \emptyset\}$. An object of $\mathbf{Nuts}$ is a pair $E = (|E|, \mathcal{T}(E))$ where $|E|$ is a set, and $\mathcal{T}(E)$ is a totality candidate on $|E|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|E|)$ [25]. And we have $t \in \mathbf{Nuts}(E, F)$ if $t \in \mathbf{Rel}(|E|, |F|)$ and $\forall u \in \mathcal{T}(E)\,(\cdot tu \in \mathcal{T}(F))$. As a direct consequence of Theorem 20, the following corollary says that the valid proofs will be interpreted as total elements.

Corollary 23. *If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}([\![\Gamma]\!])$.*

The converse of Corollary 23 is not necessarily true: there are many counterexamples, discussed in the long version.

VI. On the semantics of circular proofs

The semantics of the previous section allows us to interpret both general non-wellfounded and circular proofs, but it presents a drawback: in the case of circular proofs, the approximation semantics completely disregards the circularity of the proof objects. In the present section, we will discuss what are the challenges and how to proceed to achieve those goals. We will also see that for a fragment of circular proofs, we can use the circularity of the proof tree to define the interpretation, following Santocanale's approach.

One of the main difficulties to extend Santocanale's approach to $\mu\mathsf{LL}_\infty$ can be seen in the example of π_∞ presented in Figure 6b, page 7. Indeed, Santocanale's interpretation method strongly relies on the possibility to identify a thread by a formula, therefore π_∞ falls out of the scope of that method. Two natural options are either (i) to disregard validity in interpreting circular proofs, as we did for non-well-founded proofs in previous sections, or (ii) to constrain the validity condition to make Santocanale's method usable. We discuss the second option below by considering strongly valid proofs introduced in Section III.

A. Relating the interpretation of strongly valid proofs and their finitizations

We first want to show that the interpretation of the strongly valid circular proofs are the same as the interpretation of their finitizations in any $\mu\mathsf{LL}_\infty$ model.

Lemma 24. *Let $\vdash \Gamma^\perp, \nu XA$ be a $\mu\mathsf{LL}$ provable sequent. Then there is a unique morphism $\phi_A \in \mathcal{L}([\![\nu XA]\!], [\![I^A_\Gamma]\!])$ that satisfies the following diagram:*

$$\begin{tikzcd}
{[\![\nu XA]\!]} \arrow[r, "="] \arrow[dd, "\phi_A"'] & {[\![A]\!]([\![\nu XA]\!])} \arrow[d, "{[\![A]\!](\phi_A)}"] \\
 & {[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[d, "\mathsf{in}_1"] \\
{[\![I^A_\Gamma]\!]} \arrow[r, "="] & {[\![A]\!]([\![I^A_\Gamma]\!]) \oplus [\![\Gamma]\!]}
\end{tikzcd}$$

where I^A_Γ is the invariant formula (see Prop. 9), and in_1 is the first injection.

The proof of this lemma is detailed in the long version.

Lemma 25. *Let π be a strongly connected and strongly valid proof of $\vdash \Gamma^\perp, \nu XA$ where the last inference rule is the (ν) rule. Then $[\![\Uparrow(\pi)]\!]_\infty$ is the following morphism:*

$$[\![\Gamma]\!] \xrightarrow{[\![\pi]\!]_\infty} [\![\nu XA]\!] \simeq [\![A]\!]([\![\nu XA]\!]) \xrightarrow{[\![A]\!](\phi_A)} [\![A]\!]([\![I^A_\Gamma]\!]) \;.$$

Proof sketch. The proof which goes by induction on the structure of π and case analysis of the inference rule (r) in Definition 11 is detailed in the long version. $\square$

Theorem 26. *Let π be a strongly valid $\mu\mathsf{LL}_\infty$ proof. Then $[\![\pi]\!]_\infty = [\![\pi^{\mathsf{fin}}]\!]$, the interpretations being in any $\mu\mathsf{LL}_\infty$ model.*

Proof. We can always suppose wlog. that the conclusion of π is $\vdash \Gamma^\perp, \nu X A$. The proof is by induction on $\mathsf{size}(\pi)$. We only provide here the case that π is strongly connected, the full proof is provided in the long version.

$\triangleright$ If π is strongly connected. Then, there is an infinite path p that visits all the sequents of π. Let t be a trace of p, and, without loss of generality, let $\vdash \Gamma^\perp, \nu X A$ be the sequent where the minimal formula of t has been unfolded. Graphically, π is shown in Figure 7.

Consider now $\mu\mathsf{LL}_\infty$ proof $\Uparrow(\pi)$ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/X\right]$. The complexity of $\Uparrow(\pi)$ is strictly less than that of π, since $\mathsf{elc}(\Uparrow(\pi)) < \mathsf{elc}(\pi)$. So, by induction hypothesis, there is a $\mu\mathsf{LL}$ (finite) proof ρ of $\vdash \Gamma^\perp, A\left[I^A_\Gamma/X\right]$ such that $[\![\rho]\!] = [\![\Uparrow(\pi)]\!]$. In this case, π^{fin} is defined as in Equation (2), page 7.

Let f be the interpretation of the proof of $\vdash (I^A_\Gamma)^\perp, \nu X A$. The morphism f satisfies the following universal property:

$$\begin{tikzcd}
{[\![I^A_\Gamma]\!] = [\![A\left[I^A_\Gamma/X\right]]\!] \oplus \Gamma} \arrow[r, "{\langle \mathsf{Id}, [\![\rho]\!]\rangle}"] \arrow[d, "f"] & {[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[dl, "{[\![A]\!](f)}"] \\
{[\![\nu X A]\!] = [\![A]\!]([\![\nu X A]\!])} &
\end{tikzcd}$$

By Lemma 25, we have $[\![\rho]\!] = [\![A]\!](\phi_A) \circ [\![\pi]\!]$, and hence

$$\begin{tikzcd}
{[\![I^A_\Gamma]\!]} \arrow[r, "f"] \arrow[d, "="] & {[\![\nu X A]\!]} \arrow[dd, "="] \\
{[\![A]\!]([\![I^A_\Gamma]\!]) \oplus [\![\Gamma]\!]} \arrow[d, "{\langle \mathsf{Id}, [\![A]\!](\phi_A)\circ[\![\pi]\!]\rangle}"'] & \\
{[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[r, "{[\![A]\!](f)}"] & {[\![A]\!]([\![\nu X A]\!])}
\end{tikzcd}$$

Moreover, we have the following diagram by Lemma 24:

$$\begin{tikzcd}
{[\![\nu X A]\!]} \arrow[r, "\phi_A"] \arrow[dd, "="] & {[\![I^A_\Gamma]\!]} \arrow[d, "="] \\
& {[\![A]\!]([\![I^A_\Gamma]\!]) \oplus [\![\Gamma]\!]} \arrow[d, "{\langle \mathsf{Id}, [\![A]\!](\phi_A)\circ[\![\pi]\!]\rangle}"] \\
{[\![A]\!]([\![\nu X A]\!])} \arrow[r, "{[\![A]\!](\phi_A)}"] & {[\![A]\!]([\![I^A_\Gamma]\!])}
\end{tikzcd}$$

Hence, we have:

$$\begin{tikzcd}
{[\![\nu X A]\!]} \arrow[r, "\phi_A"] \arrow[d, "="] & {[\![I^A_\Gamma]\!]} \arrow[r, "f"] & {[\![\nu X A]\!]} \arrow[d, "="] \\
{[\![A]\!]([\![\nu X A]\!])} \arrow[r, "{[\![A]\!](\phi_A)}"] & {[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[r, "{[\![A]\!](f)}"] & {[\![A]\!]([\![\nu X A]\!])}
\end{tikzcd}$$

So, we have $[\![A]\!](f \circ \phi_A) = f \circ \phi_A$. By the universal property of $[\![\nu X A]\!]$, we conclude that $f \circ \phi_A = \mathsf{Id}$.

Since $[\![\pi^{\mathsf{fin}}]\!] = f \circ \mathsf{in}_2$, Lemma 25 ensures that:

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "{[\![\pi^{\mathsf{fin}}]\!]}"] \arrow[d, "\mathsf{in}_2"] & {[\![\nu X A]\!]} \arrow[dd, "="] \\
{[\![A]\!]([\![I^A_\Gamma]\!]) \oplus [\![\Gamma]\!] = [\![I^A_\Gamma]\!]} \arrow[d, "{\langle \mathsf{Id}, [\![A]\!](\phi_A)\circ[\![\pi]\!]\rangle}"] & \\
{[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[r, "{[\![A]\!](f)}"] & {[\![A]\!]([\![\nu X A]\!])}
\end{tikzcd}$$

As $(\langle \mathsf{Id}, [\![A]\!](\phi_A) \circ [\![\pi]\!]\rangle) \circ \mathsf{in}_2 = [\![A]\!](\phi_A) \circ [\![\pi]\!]$. Hence the following square commutes:

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "{[\![\pi^{\mathsf{fin}}]\!]}"] \arrow[d, "{[\![\pi]\!]}"] & {[\![\nu X A]\!]} \arrow[dd, "="] \\
{[\![\nu X A]\!]} \arrow[d, "{[\![A]\!](\phi_A)}"] & \\
{[\![A]\!]([\![I^A_\Gamma]\!])} \arrow[r, "{[\![A]\!](f)}"] & {[\![A]\!]([\![\nu X A]\!])}
\end{tikzcd}$$

We have $[\![A]\!](f) \circ [\![A]\!](\phi_A) = [\![A]\!](f \circ \phi_A) = \mathsf{Id}$, since $f \circ \phi_A = \mathsf{Id}$. Therefore, we conclude that $[\![\pi]\!] = [\![\pi^{\mathsf{fin}}]\!]$. $\square$

Remark 6. *Although the π^{fin} is not uniquely defined (Remark 3), the interpretation of the finitization, $[\![\pi^{\mathsf{fin}}]\!]$, is uniquely defined in any $\mu\mathsf{LL}_\infty$ model, since $[\![\pi]\!]_\infty = [\![\pi^{\mathsf{fin}}]\!]$ (Theorem 26). This also holds for any $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ models where $\mathcal{L}$ is just a $\mu\mathsf{LL}$ model, since $[\![\pi]\!]_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})} = [\![\pi]\!]_{\mathcal{L}}$ (Remark 5).*

B. Interpreting strongly valid circular proofs

Till now, we have two ways to interpret a strongly valid π:

1) In a $\mu\mathsf{LL}_\infty$ model: as we did in section IV.
2) In a $\mu\mathsf{LL}$ model: By Proposition 7, one can first finitize π, and then will interpret the finitized proof π^{fin}.

By Theorem 26, we have seen that these two interpretations are the same. In this section, we will provide a direct way to interpret π in a $\mu\mathsf{LL}$ model.

Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}$ model. We want to interpret a strongly valid proof π by induction on $\mathsf{size}(\pi)$ in $\mathcal{L}$. The general idea to interpret any valid circular proof π is first to consider two cases. If π is not strongly connected, we can always interpret it by induction on $\mathsf{size}(\pi)$. If π is strongly connected, we first choose a trace t for an infinite path p that visits all the sequents of π. Let $\nu X A$ be the minimal formula of t. We then choose a sequent $\vdash \Gamma^\perp, \nu X A$ such that the formula $\nu X A$ has been unfolded. We suppose without loss of generality that the conclusion of π is $\vdash \Gamma^\perp, \nu X A$. Graphically, π is what is described in Figure 7. We first discard all the back-edges from the leaves of π to its root, and close each of the by the same assumption F. The resulting proof, π_F, has the shape given in Figure 7. If we take a morphism $f \in \mathcal{L}([\![\Gamma]\!], [\![\nu X A]\!])$ as the interpretation of F, we have, by induction hypothesis, the interpretation of π_F as a morphism in $\mathcal{L}([\![\Gamma]\!], [\![\nu X A]\!])$. So, considering F as a parameter, one obtains from π a morphism f_π in $\mathcal{L}(C \otimes !([\![\Gamma]\!] \multimap [\![\nu X A]\!]), [\![\Gamma]\!] \multimap [\![\nu X A]\!])$ where we take C as the parameters coming from the assumptions of π.

By analyzing the proof of Theorem 26, we now want to show that the equation $f_\pi(C \otimes x) = x$ has a solution in $\mathcal{L}$, that is to say a morphism in $\mathcal{L}(\llbracket\Gamma\rrbracket, \llbracket\nu XA\rrbracket)$, denoted by $\mathsf{fix}(f_\pi)$, such that $f_\pi(C \otimes \mathsf{fix}(f_\pi)) = \mathsf{fix}(f_\pi)$. To define $\mathsf{fix}(f_\pi)$, we first consider the $\mu\mathsf{LL}_\infty$ proof $\Uparrow(\pi)$ of $\vdash \Gamma^\perp, A\left[I_\Gamma^A/X\right]$, and by induction hypothesis, we have $\llbracket\Uparrow(\pi)\rrbracket$. So, we have $\langle \mathsf{Id}, \llbracket\Uparrow(\pi)\rrbracket\rangle \in \mathcal{L}(\llbracket I_\Gamma^A\rrbracket, \llbracket A\rrbracket(\llbracket I_\Gamma^A\rrbracket))$. By the universal property of the final co-algebra $\llbracket\nu XA\rrbracket$, there is a unique morphism $f \in \mathcal{L}(\llbracket I_\Gamma^A\rrbracket, \llbracket\nu XA\rrbracket)$. Finally, we take $\mathsf{fix}(f_\pi)$ as $f \circ \mathsf{in}_2$ where $\mathsf{in}_2 \in \mathcal{L}(\llbracket\Gamma\rrbracket, \llbracket I_\Gamma^A\rrbracket)$.

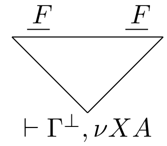


Fig. 7: π_F

As we saw, the interpretation of π, described above, depends on some choices such as choosing the validating trace t and choosing the sequent $\vdash \Gamma^\perp, \nu XA$. We do not know whether changing those parameters, we obtain the same interpretation. Nevertheless, we can prove that if those $\mu\mathsf{LL}$ models are built on top of a $\mu\mathsf{LL}_\infty$ model (as $\mathbf{Nuts}$ is for instance), then the semantics does not depend on our choice of parameters:

Theorem 27. *Let $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ be a $\mu\mathsf{LL}_\infty$ model, and π be a strongly valid proof. Then $\llbracket\pi^{\mathsf{fin}}\rrbracket_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})} = \llbracket\pi\rrbracket_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$.*

Proof. We know that $\llbracket\pi\rrbracket_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})} = \llbracket\pi\rrbracket_{\mathcal{L}}$ and $\llbracket\pi^{\mathsf{fin}}\rrbracket_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})} = \llbracket\pi^{\mathsf{fin}}\rrbracket_{\mathcal{L}}$. By Theorem 26, we have $\llbracket\pi\rrbracket_{\mathcal{L}} = \llbracket\pi^{\mathsf{fin}}\rrbracket_{\mathcal{L}}$. Hence we have $\llbracket\pi^{\mathsf{fin}}\rrbracket_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})} = \llbracket\pi\rrbracket_{\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})}$. $\square$

Thanks to Theorem 27 and Theorem 26, the interpretation function described above is well-defined for both the $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ model and the $\mu\mathsf{LL}_\infty$ model, respectively.

Remark 7. *Notice that, as it is mentioned in Remark 5, not all focused orthogonality categories are $\mu\mathsf{LL}_\infty$ models.*

Finally, as a concrete case of Theorem 27, we have the following direct corollary on the $\mathbf{Nuts}$.

Corollary 28. *If π is strongly valid, $\llbracket\pi^{\mathsf{fin}}\rrbracket_{\mathbf{Nuts}} = \llbracket\pi\rrbracket_{\mathbf{Nuts}}$.*

VII. Conclusion

In this paper, we studied the non-wellfounded proof system $\mu\mathsf{LL}_\infty$ and provided a denotational semantics of $\mu\mathsf{LL}_\infty$ in a Curry-Howard perspective.

We first gave the definition of a $\mu\mathsf{LL}_\infty$ model based on a $\mu\mathsf{LL}$ model, and proved that the semantics is preserved via a possibly infinite reduction sequence of cut-elimination rules. We studied the focused orthogonality construction to capture the syntactic validity criterion. Although the interpretation of proofs in both models of orthogonality category $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$ and $\mathcal{L}$ are the same, one can obtain more information by looking at the interpretation in $\mathcal{O}_{\perp\!\!\!\perp}(\mathcal{L})$. Benefiting from a finite representation of the circular proofs, we have provided a parameterized interpretation of strongly valid proofs in any $\mu\mathsf{LL}$ model (not only $\mu\mathsf{LL}_\infty$ models), and shown that the semantics is independent of the parameters in the case of focused orthogonality categories. We also investigated the syntactical translations between finitary and circular proofs as follows: (i) we extended Doumane's finitization results showing that, compared to [23], the wider class of strongly valid circular proofs can be finitized and (ii) we proved that the semantics is preserved both in the finitization procedure and in the translation from finitary proofs to circular ones, bringing evidence of the computational soundness of these translations.

a) Related works: Considering Santocanale & Fortier's semantics [28], [44], on the one hand, we have extended the categorical axiomatizations to treat non-wellfounded proofs in full linear logic (while they modeled only the additive fragment). On the other hand, we only benefit from the finitely presentable structure to model strongly valid circular proofs and not all valid circular proofs (but note that in the additive fragment considered by Santocanale and Fortier, every valid circular proof is strongly valid due to the simplicity of their sequent's structure). Extending this is a future work.

Das and Curzi studied in [19] the computational expressivity of functions on natural numbers in $\mu\mathsf{LJ}$ and LL, and they showed that all systems of $\mu\mathsf{LJ}$, $\mu\mathsf{MALL}$, circular $\mu\mathsf{LL}_\infty$ represent the same functions on natural numbers. However, their work is different from ours in the following two respects. First, their semantical approach is based on the computability (or realizability) semantics which consists in interpreting proofs as computational objects such as recursive functions or λ-terms and formulas with sets of such objects, while our denotational semantics interprets proofs in a mathematical domain quotienting computational invariants by this interpretation. (See [30] for further discussions on the comparison of these two traditions). Secondly, they have only treated the case of first-order function on natural numbers and do not treat other data types such as lists, streams, nor higher-order functions. Our semantics, on the contrary, makes no restrictions on the type/formulas of $\mu\mathsf{LL}$ we model.

b) Future works: A natural question consists in seeking a complete denotational model of $\mu\mathsf{LL}_\infty$ in the sense of Girard and Streicher [32], [52]. This could be useful to tackle the Brotherston-Simpson's conjecture for $\mu\mathsf{LL}$ (saying that inductive proofs and circular proofs have the same provability) as well as a *proof-relevant/denotational* version of the conjecture:

Conjecture 1 (Semantical Brotherston-Simpson's conjecture)**.** *Let π be a circular $\mu\mathsf{LL}_\infty$ proof of $\vdash \Gamma$. There exists a $\mu\mathsf{LL}$ (finite) proof π' of $\vdash \Gamma$ such that $\llbracket\pi\rrbracket = \llbracket\pi'\rrbracket$.*

We also expect to extend Theorem 27 to any $\mu\mathsf{LL}$ model:

Conjecture 2. *Let π be a strongly valid proof. Then $\llbracket\pi^{\mathsf{fin}}\rrbracket = \llbracket\pi\rrbracket$ where the interpretations of proofs are in any $\mu\mathsf{LL}$ model.*

It is not possible for a non-terminating program of type $\mathtt{nat} \multimap \mathtt{nat}$ to have a total interpretation in $\mathbf{Nuts}$. A natural (but difficult) question is whether this can be lifted to all $\mu\mathsf{LL}_\infty$ types. The same was asked by Girard for second-order types [30] almost 40 years ago; it is still an open problem.

Finally, one can notice that our proof of corollary 22 heavily relies on the syntactic results of cut-elimination and we wonder if we can obtain a direct and syntax-independent proof of Theorem 20 for bouncing validity which needs a semantical treatment for that setting.

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