Semantical Analysis of Intuitionistic Modal Logics between CK and IK

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Abstract—The intuitionistic modal logics considered between Constructive K (CK) and Intuitionistic K (IK) differ in their treatment of the possibility (diamond) connective. It was recently rediscovered that some logics between CK and IK also disagree on their diamond-free fragments, with only some remaining conservative over the standard axiomatisation of intuitionistic modal logic with necessity (box) alone. We show that relational Kripke semantics for CK can be extended with frame conditions for all axioms in the standard axiomatisation of IK, as well as other axioms previously studied. This allows us to answer open questions about the (non-)conservativity of such logics over intuitionistic modal logic without diamond. Our results are formalised using the Rocq Prover.

Index Terms—Intuitionistic modal logic, Relational semantics, Completeness, Rocq Prover

I. INTRODUCTION

Which logic provides the foundation for intuitionistic modal logics, by analogy with the logic K for classical modal logics? If we consider necessity, \Box , but disregard possibility, \diamond , then the answer has, until recently, appeared uncontroversial: we extend intuitionistic propositional logic with the inference rule of necessitation (if p is a theorem, then so is $\Box p$) and axiom

$$(\mathsf{K}_{\Box}) \quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

Since this logic, which we here call CK_{\Box} , was introduced by Božić and Došen [9], it and its extensions have been studied and applied in a literature too large to summarise here; some examples are given in Section II-A.

How CK_{\Box} should be extended with \diamond has received various answers. Consider the following axioms, where we follow the naming conventions of Dalmonte, Grellois, and Olivetti [16]:

$$\begin{array}{ll} (\mathsf{K}_{\diamond}) & \Box(\varphi \to \psi) \to (\diamond \varphi \to \diamond \psi) \\ (\mathsf{N}_{\diamond}) & \diamond \bot \to \bot \\ (\mathsf{C}_{\diamond}) & \diamond(\varphi \lor \psi) \to \diamond \varphi \lor \diamond \psi \\ (\mathsf{I}_{\diamond \Box}) & (\diamond \varphi \to \Box \psi) \to \Box(\varphi \to \psi) \end{array}$$

Constructive K (CK) [5] extends CK_{\Box} with K_{\Diamond} only; Wijesekera's K (WK) [70] extends CK further with N_{\Diamond} . These

logics are proof theoretically natural, attained by restricting the sequent calculus for classical K to single conclusions (CK) or zero or one conclusions (WK). Both were originally also motivated by applications in AI: the notion of context in knowledge representation and reasoning (CK) [22], [50], and representing states with partial knowledge, as well as constructive concurrent dynamic logic (WK) [71], as discussed further in Section II. Intuitionistic K (IK) [28]¹ has all the above axioms. It is the logic specified by Fischer Servi's translation to classical (K, S4)-bimodal logic [26]-[28], and by the standard translation to intuitionistic first order logic [66]. IK also respects the Gödel-Gentzen double negation translation from classical modal logic, although this also holds for the logic without C_{\diamond} [19]. These are by no means the only options for logics between CK and IK; we mention also Kojima's logic for intuitionistic neighbourhood models [41], which lies between CK and WK, and Forward confluence IK (FIK) [4], which modifies IK by replacing $I_{\Diamond \Box}$ with a weaker axiom.

While different notions of \diamond have arisen from different motivations, it has generally been assumed that only \diamond is controversial, and that these logics agree with CK_{\Box} on their \diamond -free fragments. This was shown to be incorrect in Grefe's 1999 thesis [34]. Grefe showed that the \diamond -free formula $(\neg\Box \bot \rightarrow \Box \bot) \rightarrow \Box \bot$ holds in IK but not in CK_{\Box} . This observation was not published, and was only recently rediscovered by Das and Marin [19], who showed, among other results, that while IK is not conservative over CK_{\Box} , the logic $CK \oplus N_{\diamond} \oplus C_{\diamond}$, and hence its sublogics such as CK and WK, are.

The (re)discovery that the logics between CK and IK are not as well understood as previously thought raises many questions. With each new axiom that is considered in this space, these questions multiply. Working via Hilbert axiomatisations only is notoriously intractable, and while proof theoretic methods were used successfully by Das and Marin to clarify the status of $CK \oplus N_{\diamond} \oplus C_{\diamond}$, the effort involved was considerable. In this paper we instead explore the (bi)relational semantics of Kripke frames. Such semantics are known for CK [50], WK [70], FIK [4], and IK [27], although imprecisions in the treatment of WK led to the soundness proof with respect to

The authors would like to gratefully acknowledge discussions with Dirk Pattinson and Alwen Tiu, and the detailed and helpful comments of the anonymous reviewers. The second author has been supported by a UKRI Future Leaders Fellowship, 'Structure vs Invariants in Proofs', project reference MR/S035540/1.

¹The name IK has been used inconsistently in the literature, sometimes for the logics that we here call CK_{\Box} and CK.

the semantics for that logic being called "inconclusive" $[50]^2$. Moreover, the different choices made both for conditions on the relations and for the interpretations of the modal operators impede comparisons between these logics.

In this paper we take the relational semantics for CK as a unifying semantics, and give frame conditions for each of the axioms N_{\diamond}, C_{\diamond}, and I_{\diamond \square}. This allows us to provide completeness proofs for each of these axioms independently. In particular, this answers the challenge of Das and Marin [19, Section 7] to provide relational semantics for logics between WK and IK. We use these semantics to analyse the \diamond -free fragments, making the new observations that CK \oplus C_{\diamond} \oplus I_{\diamond \square}, and hence CK \oplus I_{\diamond \square}, are conservative over CK_{\square}. This is a surprising result, as no logics including I_{\diamond \square} were previously shown to retain conservativity; it is now clear that the *combination* of I_{\diamond \square} and N_{\diamond} is to blame here.

We formalise all our results in the Rocq Prover [68], which not only adds confidence to our results (in particular, the doubt raised [50] about the relational semantics for WK may now be considered settled), but is a crucial working tool for managing the profusion of logics which arise as one considers new axioms. As a proof of concept of this methodology of working from a base relational semantics for CK with support from Rocq, we go on to provide relational semantics and conservativity results for Kojima's logic, and for the weakening of $I_{\bigcirc \square}$ used in FIK. Each mechanised result in the paper is accompanied by a clickable rooster symbol " $\ref{eq:accompanied}$ " leading to its mechanisation. The full mechanisation can be found at https://github.com/ianshil/CK and its documentation at https://ianshil.github.io/CK/toc.html.

This paper begins by discussing constructive modal logics in more depth in Section II, before introducing the basic syntax in Section III. We give sound and strongly complete relational semantics for our base logic CK in Section IV, then give frame conditions and completeness results for N_{\diamond} , C_{\diamond} and $I_{\diamond\Box}$ in Sections V and VI, and compare the resulting logics' \diamond -free fragments in Section VII. We extend our techniques to other axioms from the literature in Section VIII. We finish by discussing our Rocq formalisation in Section IX and surveying possible further work in Section X.

II. CONSTRUCTIVE K AND ITS EXTENSIONS

In this section we elaborate on the logics that lie at the heart of the paper.

A. Intuitionistic logic with boxes

Intuitionistic modal logics with a necessity (box) modality but no diamond, or where diamond is viewed as a derived modality, date back to 1965 [12]. Early literature often takes an S4-perspective on the modality [6], [29], [53], [58]. The first occurrence of CK_{\Box} appears to be in 1984 [9], [24], where it is called $HK\Box$. Subsequently, it has been widely studied under various names, including IntK [74], IntK_{\Box} [73], [75], IK_{\Box} [5], IK [39] and iK [19]. We highlight some of its appearances.

Example II.1 (Modalities for context). The modal operator \Box can be used to formalise the idea of a *context*, a notion in the field of *knowledge representation*. For example, if \Box denotes the context of Sherlock Holmes, then it is true that Sherlock Holmes lives in Baker Street, i.e. \Box (Sherlock lives on Baker Street). We can use multiple modalities, denoted as \Box_{κ} or $ist(\kappa, \varphi)$ (for *is t*rue), to model several contexts κ .

From a computer science point of view, contexts can for example be used to deal with databases with multiple conventions [48], [49]. More generally McCartey states that an "AI goal" is to allow simple axioms for commonsense to be lifted to other contexts [47], [49]. This idea was further studied in e.g. [14], [45], [52], and in [22] it was shown that the common core of the latter three is given by (a multimodal version of) CK_{\Box} .

Example II.2 (Modalities for knowledge). An epistemic interpretation of $\Box \varphi$ is that an agent *knows* or *believes* φ to be true. In an intuitionistic context, the epistemic operator can be used to model an ideal reasoner (the agent) in a growing informational state (an intuitionistic Kripke frame) [38], [60]. This motivates the reflection principle $\Box \varphi \rightarrow \varphi$: if an agent knows that φ is true, then it is true.

Alternatively, one can take $\Box \varphi$ to represent "belief and knowledge as the product of verification" [3]. In this view, the intuitionistic truth of a proposition entails knowledge of it, because an intuitionistic proof is a verification, so one gets the co-reflection principle $\varphi \rightarrow \Box \varphi$ as an axiom. A priori, this logic does not rule out false beliefs. The extension of CK_{\Box} with co-reflection is called IEL⁻, and has recently received a lot of attention [10], [61], [62], [67]. (Incidentally, IEL⁻ coincides with the inhabitation logic of Haskell's applicative functors [46], as was noted in [44], [61].)

Example II.3 (Curry-Howard correspondence). Constructive versions of S4 received a lot of attention from a type-theoretic perspective [2], [7], [32]. This sparked attempts to give a Curry-Howard correspondence for CK_{\Box} as well. The first such correspondence was established by Bellin, De Paiva and Ritter [5], and was later refined by Kakutani [39]. In their work, \Box is the type former corresponding to a term constructor which can be interpreted as a sort of substitution. This is still an active field of research: a new correspondence for the $\land\lor$ -free fragment of CK_{\Box} was recently discovered by Acclavio, Catta and Olimpieri [1], and a Curry-Howard correspondence for IEL⁻ was given in [10], [62].

Extensions of CK_{\Box} , for example with the S4 axioms, have also found many applications, ranging from hardware verification [25] to access control [31] to staged computation [20], [21], [51], and from the productivity of recursive definitions [8] to global elements in synthetic topology [65].

²More precisely, Mendler and De Paiva argued that flaws in Wijesekera's work with WK made it unsuitable to conclude soundness for CK, but their argument holds equal force as a criticism of the development for WK itself.

B. Intuitionistic logic with boxes and diamonds

As in the mono-modal case, the study of intuitionistic modal logic with two modalities, \Box and \diamond , started with intuitionistic analogues of S4, for example in [11], [13], [55], [58], [59]. These were then generalised to intuitionistic counterparts of K, where the variety of axioms defining \diamond and relating \Box and \diamond (such as N $_{\diamond}$, C $_{\diamond}$ and I $_{\diamond\Box}$) resulted in a wide variety of intuitionistic modal logics.

One of the simplest intuitionistic modal logics with an independent box and diamond modality is Constructive K (CK). This extends CK_{\Box} with K_{\Diamond} , and was described in [5], following an adaptation of Prawitz's suggestions [58] for intuitionistic S4 to K. Adding various configurations of N_{\Diamond} , C_{\Diamond} and $I_{\Diamond\Box}$ gives rise to logics including WK [70], [71], and IK [27], [28], [57], [66]. We point out some uses of these logics.

Example II.4 (Satisfiability in context). In the setting of knowledge representation, $\diamond_{\kappa}\varphi$ can be can be interpreted as φ being *satisfiable in context* κ [50]. Under this light, the diamond-containing axiom K_{\diamond} of CK is a sensible one to adopt. Indeed, truth of the implication of $\varphi \rightarrow \psi$ in a given context allows one to infer the satisfiability of ψ from satisfiability of φ . However, we may not so readily accept other axioms, like N_{\diamond} and C_{\diamond}. For example, N_{\diamond} declares that falsity is satisfiable in no context, so adding it to our system prevents us from identifying inconsistent contexts.

Example II.5 (Parallel computation). The logic WK is obtained by adding N_{\diamond} to CK [70], [71]. It was put forward as a constructivised version of concurrent dynamic logic [54]. Here $\diamond_{\alpha}\varphi$ means that an execution of program α reaches a state where φ holds, and $\diamond_{\alpha\cap\beta}$ is read as " α and β can be executed in parallel so that upon termination (in either computation path) φ holds," so it is equivalent to $\diamond_{\alpha}\varphi \wedge \diamond_{\beta}\varphi$. This interpretation prevents distributivity of diamonds over joins (i.e. C_{\diamond}), because the truth of $\diamond_{\alpha\cap\beta}(\varphi \lor \psi)$ may be witnessed by $\diamond_{\alpha}\varphi$ and $\diamond_{\beta}\psi$.

Example II.6 (Diamonds for consistency). Both classically [37] and intuitionisticially [72], the diamond operator is used in epistemic logic to denote consistency or a kind of possibility of φ with respect to an agent's knowledge. The disentanglement of box and diamond in the intuitionistic setting allows us to reevaluate the axioms we impose on diamonds.

For example, if $\varphi \to \psi$ is known and φ is possible (or consistent), then it stands to reason that ψ is consistent too, so K_{\Diamond} is a plausible axiom. The axiom N_{\Diamond} holds in the intuitionistic epistemic logic studied in [72], but we may not always want this to be the case: Since the point of view taken in [3] allows an agent to hold a false belief, \bot could be a consequence of their knowledge, so that $\Diamond \bot$ holds and we must reject N_{\Diamond} .

Example II.7 (Evaluation logic). In [56], Pitts introduces *evaluation logic*, which is an extension of IK. This logic has modal formulas of the form $[x \leftarrow E]\varphi(x)$ and $\langle x \leftarrow E \rangle \varphi(x)$,

which express that if x is evaluated to E, then $\varphi(x)$ will necessarily or possibly hold. The logic is designed to reason about computation specified using a style of operational semantics called natural semantics.

Example II.8 (Curry-Howard correspondence). It is natural to wonder whether the Curry-Howard correspondence for CK_{\Box} can be extended to one of the above-mentioned constructive modal logics with a diamond. A correspondence for CK was given in [5], but this turned out to have deficiencies [23], [39], [40] which as of yet have not been entirely corrected.

A correspondence for WK would be particularly attractive, given its interpretation as parallel computation. This would allow one to generate programs containing concurrency which are verified by extraction.

III. THE FORMAL SYSTEM(S)

In this section we fix the syntax and axiomatic calculus for CK and its extensions. Taking a countably infinite set of propositional variables $Prop = \{p, q, r, ...\}$, we define the language L via the following grammar (P):

$$\varphi ::= p \in \operatorname{Prop} |\perp |\varphi \land \varphi | \varphi \lor \varphi | \varphi \to \varphi | \Box \varphi | \Diamond \varphi$$

We abbreviate $\neg \varphi := \varphi \rightarrow \bot$ and $\top := \neg \bot$. We use Greek lowercase letters, e.g. φ, ψ, χ and δ , to denote formulas, and Greek uppercase letters, e.g. $\Gamma, \Delta, \Phi, \Psi$, for multisets of formulas. For such a multiset Γ we define the multisets $\Box(\Gamma) := \{\Box \varphi \mid \varphi \in \Gamma\}$ and $\Box^{-1}(\Gamma) := \{\varphi \mid \Box \varphi \in \Gamma\}$ and similarly for $\Diamond(\Gamma)$ and $\Diamond^{-1}(\Gamma)$. If Γ is finite, $\bigvee \Gamma$ denotes the disjunction of all formulas in Γ . We distinguish the logical connectives in **L** from those used in our metalogic with a dot on top of the metalogical connectives, e.g. \neg . Since Prop is countably infinite and we have finitely many connectives we can enumerate the formulas of **L** (\supseteq).

All logics we consider are syntactically defined as extensions of the base logic CK with axioms. We describe this formally by defining a logic parametrised in a set $Ax \subseteq L$ of axioms, so that $Ax = \emptyset$ corresponds to CK.³ We denote by $\mathcal{I}(Ax)$ the set of all instances of axioms in a given set Ax.

Definition III.1 (A). Let CKAx (A) be an axiomatisation of intuitionistic logic (A) together with K_{\Box} and K_{\Diamond} . For a set $Ax \subseteq L$, define the generalised Hilbert calculus CK \oplus Ax by:

$$\begin{array}{ll} (\mathsf{Ax}) & \frac{\varphi \in \mathcal{I}(\mathsf{CKAx}) \cup \mathcal{I}(\mathsf{Ax})}{\Gamma \vdash \varphi} & (\mathsf{Nec}) & \frac{\emptyset \vdash \varphi}{\Gamma \vdash \Box \varphi} \\ (\mathsf{MP}) & \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \rightarrow \psi}{\Gamma \vdash \psi} & (\mathsf{El}) & \frac{\varphi \in \Gamma}{\Gamma \vdash \varphi} \end{array}$$

We call *consecutions* expressions of the form $\Gamma \vdash \varphi$. We say that $\Gamma \vdash \varphi$ is *provable in* $\mathsf{CK} \oplus \mathsf{Ax}$, and write $\Gamma \vdash_{\mathsf{Ax}} \varphi$, if there exists a tree of consecutions built using the rules above with $\Gamma \vdash \varphi$ as root and adequate applications of rules $\mathsf{E}I$ and Ax as leaves. We also write $\Gamma \not\vdash_{\mathsf{Ax}} \varphi$ if $\neg (\Gamma \vdash_{\mathsf{Ax}} \varphi)$, and

 $^{^{3}}$ In the formalisation we use as parameter a set of formulas closed under substitution. Given a set Ax of axioms, the set of all instances of axioms in Ax is such a set.

write $\Gamma \vdash_{A_X} \Delta$ for $\Delta \subseteq \mathbf{L}$ if there is a finite $\Delta' \subseteq \Delta$ such that $\Gamma \vdash_{A_X} \bigvee \Delta'$. If $A_X = \{A_0, \dots, A_n\}$ is finite, we write $\mathsf{CK} \oplus A_0 \oplus \dots \oplus A_n$ for $\mathsf{CK} \oplus \mathsf{Ax}$.

Sometimes $CK \oplus Ax$ has an existing name in the literature. For example, $CK \oplus N_{\diamond}$ is known as WK. In such cases, we use both names interchangeably. The rules displayed below, where σ is a uniform substitution, are admissible in $CK \oplus Ax$ for any set of axioms Ax:

$$\frac{\Gamma \vdash \varphi}{\Gamma, \Gamma' \vdash \varphi} \quad \frac{\{\Gamma \vdash \delta \mid \delta \in \Delta\} \quad \Delta \vdash \varphi}{\Gamma \vdash \varphi} \quad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi^{\sigma}}$$
$$\frac{\Gamma, \varphi \vdash \psi}{\overline{\Gamma \vdash \varphi \rightarrow \psi}} \quad \frac{\Gamma \vdash \varphi}{\Box(\Gamma) \vdash \Box \varphi}$$

The three topmost rules show that $\mathsf{CK} \oplus \mathsf{Ax}$ is a monotone (P), compositional (P) and structural (P) relation, respectively. Furthermore, we can show that $\Gamma \vdash_{\mathsf{Ax}} \varphi$ if and only if there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathsf{Ax}} \varphi$ (P). Therefore $\mathsf{CK} \oplus \mathsf{Ax}$ is a finitary logic [42]. The left rule of the bottom row can be applied in both directions and corresponds to the deduction-detachment theorem (P, P).⁴ The right rule of the bottom row captures the modal sequent calculus rule (P).

Definition III.2. A set of formulas $\Gamma \subseteq \mathbf{L}$ is a *theory* (\clubsuit) if it is deductively closed, i.e. $\Gamma \vdash_{\mathsf{Ax}} \varphi$ implies $\varphi \in \Gamma$. It is *prime* (\clubsuit) if $\varphi \lor \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$, for all $\varphi, \psi \in \mathbf{L}$.

We note that the prime theories we consider need not be consistent, and can thus contain \perp . This reflects the existence of an inconsistent world in the semantics, defined in Section IV.

Lemma III.3 (Lindenbaum **P**). Let $\Gamma \cup \Delta \subseteq \mathbf{L}$. If $\Gamma \not\vdash_{\mathsf{Ax}} \Delta$ then there is a prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \not\vdash_{\mathsf{Ax}} \Delta$.

This can be proved by a routine argument. We often use the Lindenbaum lemma with Δ of the form $\{\varphi\}$ (P).

IV. RELATIONAL SEMANTICS FOR CK

We present a relational semantics for CK which is a light modification of Mendler and De Paiva [50]. Their semantics is characterised by the interpretation of *both* modalities over all intuitionistic successors of worlds, and by the existence of worlds that satisfy all formulas, including \perp . Such worlds were introduced by Veldman [69] as "sick" worlds, whereas Mendler and De Paiva call them "fallible"; we follow Ilik, Lee and Herbelin's *exploding* terminology [36]. Because all exploding worlds are essentially the same with respect to formula satisfaction we slightly simplify the Mendler-De Paiva semantics by using a single exploding world, instead of a set.

Definition IV.1. A CK-*frame* (\clubsuit) is a tuple $(X, \mathfrak{G}, \leq, R)$ where (X, \leq) is a preorder, $\mathfrak{G} \in X$ is a maximal element

of (X, \leq) , and R is a binary relation on X such that $\mathfrak{G}Rx$ if and only if $x = \mathfrak{G}$. We denote by \mathcal{CK} the class of all CK-frames.

A valuation is a map V that assigns to each proposition letter p an upset V(p) of (X, \leq) such that $\mathfrak{G} \in V(p)$. A CKmodel (\mathfrak{P}) is a CK-frame with a valuation. The interpretation of a formula φ at a world x in a CK-model $\mathfrak{M} = (X, \mathfrak{G}, \leq$, R, V) (\mathfrak{P}) is defined recursively by

Let $\Gamma \cup \{\varphi\} \subseteq \mathbf{L}$ and let \mathfrak{M} be a CK-model. We write $\mathfrak{M}, x \Vdash \Gamma$ if x satisfies all $\psi \in \Gamma$, and we say that \mathfrak{M} validates $\Gamma \vdash \varphi$ if $\mathfrak{M}, x \Vdash \Gamma$ implies $\mathfrak{M}, x \Vdash \varphi$ for all worlds x in \mathfrak{M} . A CKframe \mathfrak{X} validates $\Gamma \vdash \varphi$ if every model of the form (\mathfrak{X}, V) validates the consecution, and it validates a formula φ if it validates the consecution $\emptyset \vdash \varphi$. If \mathcal{F} is a class of CK-frames, then we say that Γ semantically entails φ on \mathcal{F} (\mathfrak{P}), and write $\Gamma \Vdash_{\mathcal{F}} \varphi$, if every CK-frame in \mathcal{F} validates $\Gamma \vdash \varphi$.

The universal quantifier in the interpretation of \diamond prevents distributivity of diamond over disjunctions, and thus is often not necessary when studying logics that include C_{\diamond} . We reiterate that \bullet is a maximal element in (X, \leq) but not necessarily a top element. That is, there are no elements above \bullet in the partial order \leq (other than \bullet itself), but \bullet does not necessarily lie above all elements of X.

We will show that CK-frames form a sound and complete semantics for CK.

Lemma IV.2 (Persistence **P**). If $\mathfrak{M}, x \Vdash \varphi$ and $x \leq y$ then $\mathfrak{M}, y \Vdash \varphi$.

Proof. By induction on the structure of φ . The \perp case holds because $\mathbf{\bullet}$ is maximal. All other cases are as usual.

Proposition IV.3 (Soundness **P**). If $\Gamma \vdash_{\mathsf{CK}} \varphi$, then $\Gamma \Vdash_{\mathcal{CK}} \varphi$.

Proof. By routine induction on the structure of a proof of $\Gamma \vdash \varphi$. In particular, validity of (any instance of) the axiom $\bot \rightarrow p$ follows from maximality of \mathfrak{G} .

Next, we define a canonical model for the logic $CK \oplus Ax$, where Ax is any set of axioms. This gives rise to a CK-model that validates precisely the consecutions derivable in $CK \oplus Ax$. We can use this to obtain completeness for a specific logic $CK \oplus Ax'$ with respect to some class \mathcal{F} of CK-frames by showing that all frames in \mathcal{F} validate the axioms in Ax', and

⁴This notably implies that CK and WK satisfy the deduction theorem. Mendler and De Paiva make the opposite claim [50, Footnote 2] in their analysis of Wijesekera's work, but despite some imprecision in his definitions, we believe that Wijesekera had in mind a calculus like ours where the rule (Nec) has \emptyset , and not a general Γ , in its premise.

that the CK-frame underlying the canonical model is in \mathcal{F} . In order to achieve the latter we sometimes have to modify the canonical model construction, as we will see in Section VI.

While canonical models are often based on *theories* (Definition III.2), we adapt Wijesekera's use of *segments* [70]. These are theories paired with a set of theories that intuitively denote their modal successors. This technique prevents distributivity of diamonds over disjunctions, corresponding to the C_{\diamond} axiom. Because we have an exploding world, the theories we use to define our segments are allowed to contain \perp .

Definition IV.4. A segment (\mathbf{P}) is a pair (Γ, U) where Γ is a prime theory and U is a set of prime theories such that:

1) if $\Box \varphi \in \Gamma$ then $\varphi \in \Delta$ for all $\Delta \in U$;

2) if $\Diamond \varphi \in \Gamma$ then $\varphi \in \Delta$ for some $\Delta \in U$.

Write SEG for the set of all segments and define relations \subseteq (P) and R (P) on SEG by:

$$\begin{aligned} (\Gamma, U) & \subseteq (\Gamma', U') \quad \text{iff} \quad \Gamma \subseteq \Gamma' \\ (\Gamma, U) R(\Gamma', U') \quad \text{iff} \quad \Gamma' \in U \end{aligned}$$

Note that \subseteq defines a preorder on SEG. Furthermore, observe that $(\mathbf{L}, \{\mathbf{L}\})$ is a segment, and conversely any segment of the form (\mathbf{L}, U) must have $U = \{\mathbf{L}\}$: $\diamond \perp \in \mathbf{L}$ implies that U is non-empty, and $\Box \perp \in \mathbf{L}$ implies that each of its elements is \mathbf{L} . Thus, setting $\mathfrak{O} = (\mathbf{L}, \{\mathbf{L}\})$ (**p**) gives rise to a CK-frame $\mathfrak{X}_{\mathsf{CK}\oplus\mathsf{Ax}} := (SEG, \mathfrak{O}, \subseteq, R)$ (**p**). We can equip this frame with the valuation V given by $V(p) = \{(\Gamma, U) \in SEG \mid p \in \Gamma\}$ for all $p \in \text{Prop}$ (**p**). Then we obtain the model $\mathfrak{M}_{\mathsf{CK}\oplus\mathsf{Ax}} = (\mathfrak{X}_{\mathsf{CK}\oplus\mathsf{Ax}}, V_{\Sigma})$ (**p**).

Lemma IV.5 (P). Let Γ be a prime theory such that $\Diamond \varphi \notin \Gamma$. Then there exists a segment (Γ, U) such that for all $\Delta \in U$ we have $\varphi \notin \Delta$.

Proof. For any $\Diamond \theta \in \Gamma$, we have $\Box^{-1}(\Gamma), \theta \not\vdash_{A_{X}} \varphi$, for otherwise we would get $\Gamma, \Diamond \theta \vdash_{A_{X}} \Diamond \varphi$ hence $\Diamond \varphi \in \Gamma$. Now use the Lindenbaum lemma III.3 to find a prime theory Δ_{θ} containing $\Box^{-1}(\Gamma)$ and θ but not φ . Then $(\Gamma, \{\Delta_{\theta} \mid \Diamond \theta \in \Gamma\})$ is a segment with the desired property. \Box

In particular, the previous lemma implies that for each prime theory Γ we can construct a Σ -segment of the form (Γ, U) : if $\Diamond \perp \notin \Gamma$ we use Lemma IV.5 and if $\Diamond \perp \in \Gamma$ then we can take U to be the set of prime theories containing $\Box^{-1}(\Gamma)$.

Lemma IV.6 (Truth lemma **)**. For any segment (Γ, U) and formula $\varphi \in \mathbf{L}$ we have $(\Gamma, U) \Vdash \varphi$ iff $\varphi \in \Gamma$.

Proof. By induction on the structure of φ . The cases for proposition letters and \bot hold by construction. The inductive steps for meets, joins and implications are routine.

If $\varphi = \Diamond \psi$ then by construction $\Diamond \psi \in \Gamma$ implies $(\Gamma, U) \Vdash \Diamond \psi$. Conversely, if $\Diamond \psi \notin \Gamma$ then using Lemma IV.5 we can find a Σ -segment (Γ, U') such that $\psi \notin \Delta$ for all $\Delta \in U'$. Since $(\Gamma, U) \subseteq (\Gamma, U')$ we have $(\Gamma, U) \nvDash \Diamond \psi$ by persistence.

Lastly, if $\varphi = \Box \psi$ and $\Box \psi \in \Gamma$ then by construction we have $(\Gamma, U) \Vdash \Box \psi$. For the converse, suppose $\Box \psi \notin \Gamma$. Then $\Box^{-1}(\Gamma) \not\vdash_{\mathsf{Ax}} \psi$ (for otherwise we would have $\Box \psi \in \Gamma$), so

we can use the Lindenbaum lemma to find a prime theory Γ_{ψ} containing $\Box^{-1}(\Gamma)$ but not ψ . Now we have that $(\Gamma, U \cup \{\Gamma_{\psi}\})$ is a Σ -segment and $(\Gamma, U) \subseteq (\Gamma, U \cup \{\Gamma_{\psi}\})$, so that Γ_{ψ} witnesses the fact that $(\Gamma, U) \not\vDash \Box \psi$. \Box

Theorem IV.7 (Strong completeness **P**). Let \mathcal{F} be a class of frames such that $\mathfrak{X}_{\mathsf{CK}\oplus\mathsf{A}_{\mathsf{X}}} \in \mathcal{F}$, and every $\mathfrak{X} \in \mathcal{F}$ validates Ax. Then, $\Gamma \Vdash_{\mathcal{F}} \varphi$ entails $\Gamma \vdash_{\mathsf{A}_{\mathsf{X}}} \varphi$.

Proof. We reason by contrapositive. Suppose $\Gamma \not\vdash_{A_{\mathsf{X}}} \varphi$. Then we can find a prime theory Γ' containing Γ but not φ , and extend Γ to a segment of the form (Γ', U) (\nearrow). The truth lemma implies $(\Gamma', U) \Vdash \chi$ for all $\chi \in \Gamma$ while $(\Gamma', U) \not\vdash \varphi$. Since $\mathfrak{X}_{\mathsf{CK} \oplus \mathsf{A}_{\mathsf{X}}} \in \mathcal{F}$ by assumption, we find $\Gamma \not\Vdash_{\mathcal{F}} \varphi$.

Remark IV.8. The canonical model construction can also be performed relative to a finite set Σ of formulas. This gives rise to a finite canonical model, a truth lemma relative to Σ , and ultimately a finite model property. Since this is beyond the scope of this paper we omit the details.

As the class of frames CK vacuously validates all additional axioms of CK, i.e. none, and the frame $\mathfrak{X}_{CK} \in CK$, we exploit the result above to obtain strong completeness for CK.

Theorem IV.9 (Strong completeness for CK **)**. *If* $\Gamma \Vdash_{CK} \varphi$ *then* $\Gamma \vdash_{CK} \varphi$.

Remark IV.10. Wijesekera [70] uses a similar construction as above to obtain completeness for WK. Besides incorporating an inconsistent world, the main difference is that our canonical model is based on the set of all segments, while Wijesekera uses recursion to generate a model from a given segment. We also note that Wijesekera claims completeness with respect to partially ordered frames, but their canonical model construction only gives a preorder. The claim for partially ordered frames can be recovered via an unravelling construction akin to [15, Section 3.3].

V. THREE AXIOMS BETWEEN CK AND IK

The logic IK can be obtained by extending CK with N_{\diamond} , C_{\diamond} and $I_{\diamond\square}$. This section examines these three axioms individually. We give frame conditions that guarantee validity for each of them, and then refine these to frame correspondence conditions. While the latter conditions provide sound and strongly complete semantics for the extension of CK with any combination of N_{\diamond} , C_{\diamond} and $I_{\diamond\square}$, we use the former when possible because of their greater simplicity. In particular, we provide the birelational semantics for CK \oplus $N_{\diamond} \oplus C_{\diamond}$ called for by Das and Marin [19, Section 7].

Definition V.1. Let $\mathfrak{X} = (X, \mathfrak{C}, \leq, R)$ be a CK-frame. We identify three frame conditions $(\mathbb{P}, \mathbb{P}, \mathbb{P})$:

(
$$\mathbb{N}_{\diamond}$$
-suff) $\forall x \ (xR \bullet \text{ implies } x = \bullet)$

(C_{\diamond}-suff) $\forall x \exists x' (x \leq x' \text{ and } \forall y, z \text{ (if } x \leq y \text{ and } x'Rz \text{ then } \exists w \in X \text{ s.t. } yRw \text{ and } z \leq w))$

$$\begin{array}{ll} \forall x, y, z \text{ s.t } xRy \leq z \ (\exists u \in X \text{ s.t. } x \leq uRz \\ \text{and } \forall s \in X \text{ s.t. } u \leq s, \exists t \in X \text{ s.t. } sRt \text{ and } z \leq t \end{array}$$

Proposition V.2. Let $\mathfrak{X} = (X, \mathfrak{G}, \leq, R)$ be a frame and $A \in \{\mathbb{N}_{\diamond}, \mathbb{C}_{\diamond}, \mathbb{I}_{\diamond \square}\}$. If \mathfrak{X} satisfies (A-suff), then \mathfrak{X} validates A.

Proof. This result $(\mathbf{p}, \mathbf{p}, \mathbf{p})$ follows from the fact that each of the conditions above implies the correspondence condition of the axiom under consideration $(\mathbf{p}, \mathbf{p}, \mathbf{p})$, given below.

Remark V.3. If we take x = x' in (C_{\diamond}-suff), we obtain a stronger condition (**P**),

(C_{$$\diamond$$}-strong) $\forall x, y, z (\text{if } x \leq y \text{ and } xRz \\ \text{then } \exists w (yRw \text{ and } z \leq w)).$

This is a standard frame condition for the semantics of IK [27], [57], [66]. It implies that we can ignore the universal quantifier in the interpretation of $\Diamond \varphi$, looking only at modal successors of the current world. In presence of (C_{\diamond}-strong), condition (I_{$\diamond \Box$}-suff) is equivalent to (\clubsuit)

 $(I_{\Diamond \Box}$ -weak) $\forall x, z, u \text{ (if } xRz \leq u \text{ then } \exists y (x \leq yRu)),$

which is also used in the standard semantics of IK.



Fig. 1: Sufficient conditions for validity of N_{\diamond} , C_{\diamond} and $I_{\diamond\Box}$. Unlabelled arrows denote the intuitionistic relation. Solid arrows indicate universally quantified relations, while the dashed ones indicate existential ones.

The five frame conditions introduced in Definition V.1 and Remark V.3 are depicted in Figure 1. While these are sufficient to ensure validity of certain axioms, none of them are necessary. The next examples illustrate this for (N_{\diamond} -suff) and ($I_{\diamond\Box}$ -suff).

Example V.4. Consider the frame depicted in Figure 2a. This validates $\Diamond \perp \rightarrow \perp$, because $\mathbf{\sigma}$ is the only world that satisfies $\Diamond \perp$. But it does not satisfy (N_{\diamond}-suff), because $xR\mathbf{\sigma}$.

Example V.5. Consider the frame given in Figure 2b. This satisfies neither $(I_{\Diamond \Box}$ -suff) nor $(I_{\Diamond \Box}$ -weak). However, the frame does validate $I_{\Diamond \Box}$. To see this, we show that every world that satisfies $\Diamond p \rightarrow \Box q$ also satisfies $\Box (p \rightarrow q)$. For u, v, w this follows immediately from their lack of modal successors, which implies that they trivially satisfy $\Box (p \rightarrow q)$. For y it

follows from the fact that $y \Vdash \Diamond p \to \Box q$ implies that either $w \nvDash p$ or $w \Vdash q$, so that $w \Vdash p \to q$ whence $y \Vdash \Box (p \to q)$. Lastly, suppose $x \Vdash \Diamond p \to \Box q$. If none of u, v, w satisfy p then they all satisfy $p \to q$, and hence $x \Vdash \Box (p \to q)$. If any of u, v, w satisfy p then so does w, which implies $x \Vdash \Diamond p$. Then we have $x \Vdash \Box q$, so that $u \Vdash q$ and hence $u, v, w \Vdash q$. Thus $u, v, w \Vdash p \to q$, which again implies $x \Vdash \Box (p \to q)$.



Fig. 2: Frames witnessing non-necessity of some of the sufficient conditions from Definition V.1 and Remark V.3.

We now give exact correspondence conditions for each of the axioms N_{\diamond} , C_{\diamond} and $I_{\diamond\Box}$. Two of the three correspondence conditions are depicted in Figure 3 below.

Proposition V.6 (P). A CK-frame $\mathfrak{X} = (X, \mathfrak{G}, \leq, R)$ validates \mathbb{N}_{\diamond} if and only if it satisfies:

(
$$\mathbb{N}_{\diamond}$$
-corr) $\forall x \ (if \ y R \bullet for \ all \ y \ge x, \ then \ x = \bullet)$

Proof. The correspondence condition implies validity of N_{\diamond} by definition. Conversely, if the correspondence condition does not hold then this must be witnessed by a world that satisfies $\diamond \perp$ but not \perp .

To simplify the statement of the correspondence condition for C_{\diamond} we use the following notation: if (X, \leq) is a preorder, R a binary relation on $X, x \in X$ and $a \subseteq X$, then $\downarrow a := \{x \in X \mid x \leq y \text{ for some } y \in a\}$ denotes the downset generated by $a, R[x] := \{y \in X \mid xRy\}$ and $R^{-1}(x) = \{y \in X \mid yRx\}$.

Proposition V.7 (P). A frame $\mathfrak{X} = (X, \mathfrak{G}, \leq, R)$ validates C_{\diamond} if and only if it satisfies:

(C_o-corr)
$$\forall x, y, z \text{ (if } y, z \notin R^{-1}(\mathbf{\bullet}) \text{ and } x \leq y \text{ and } x \leq z$$

then $\exists w (x \leq w \text{ and } R[w] \subseteq \downarrow R[y]$
and $R[w] \subseteq \downarrow R[z]))$

Proof. Suppose \mathfrak{X} satisfies (\mathbb{C}_{\diamond} -corr). Let x be a world, V any valuation and suppose $x \not\models \Diamond p \lor \Diamond q$. Then $x \not\models \Diamond p$ and $x \not\models \Diamond q$, so there exist $y \ge x$ and $z \ge x$ such that $R[y] \cap V(p) = \emptyset$ and $R[z] \cap V(q) = \emptyset$. We must have $y, z \notin R^{-1}(\mathbf{C})$, so by (\mathbb{C}_{\diamond} -corr) we get some $w \ge x$ such that every R-successor of w lies below an R-successor of y and below an R-successor of z. This implies that $R[w] \cap (V(p) \cup V(q)) = \emptyset$, so that w witnesses $x \not\models \Diamond(p \lor q)$. Since this holds for every $x \in X$ and every valuation, it follows that $\mathfrak{X} \Vdash \Diamond(p \lor q) \to \Diamond p \lor \Diamond q$.

For the converse, suppose that $(C_{\diamond}$ -corr) does not hold. Then we can find x, y, z satisfying $y, z \notin R^{-1}(\mathbf{G}^{\circ})$ and $x \leq y$ and $x \leq z$, and such that there is no $w \geq x$ such that every R-successor of w lies below R-successors of y and z. Taking $V(p) = (X \setminus \downarrow R[y]) \cup \{ \mathfrak{C} \}$ and $V(q) = (X \setminus \downarrow R[z]) \cup \{ \mathfrak{C} \}$ then results in a model where x satisfies $\Diamond(p \lor q)$ but not $\Diamond p$ or $\Diamond q$, whence $x \not\models \Diamond(p \lor q) \rightarrow \Diamond p \lor \Diamond q$. \Box

Proposition V.8 (A). A frame $\mathfrak{X} = (X, \mathfrak{G}, \leq, R)$ validates $I_{\Diamond \Box}$ if and only if it satisfies:

$$\begin{array}{ll} (I_{\Diamond\square}\text{-corr}) & \forall x, y, z (\textit{if } xRy \leq z \neq \texttt{@} \\ & \textit{then } \exists u, w \ (x \leq uRw \leq z \\ & \textit{and } \forall s \ (\textit{if } u \leq s \ \textit{then } sR\texttt{@} \\ & \textit{or } \exists t (sRt \ \textit{and } z \leq t)))) \end{array}$$

Proof. Suppose \mathfrak{X} satisfies $(| \diamond_{\Box} \text{-corr})$. Let V be any valuation and suppose $x' \not\Vdash \Box(p \to q)$. Then there exist worlds x, yand z such that $x' \leq xRy \leq z$ and $z \Vdash p$ and $z \not\Vdash q$. Since $z \not\Vdash q$ we must have $z \neq \mathfrak{C}$, so we can find u, w with the properties mentioned in $(|_{\diamond_{\Box}} \text{-corr})$. Then $u \Vdash \diamond_{P}$ because each intuitionistic successor s of u can modally see \mathfrak{C} or some successor of z, both of which satisfy p. But we have $u \not\Vdash \Box q$, because $u \leq uRw$ and $w \leq z$, hence $w \not\nvDash q$. Since $x' \leq x \leq u$, we conclude $x' \not\Vdash \diamond p \to \Box q$.

For the converse, suppose that the frame validates ($\Diamond p \rightarrow$ $\Box q) \rightarrow \Box (p \rightarrow q)$ and let $x, y, z \in X$ be such that $xRy \leq z$ z and $z \neq \mathbf{0}$. Since the formula is valid, it holds for all valuations and we can exploit its truth under any valuation that suits us. Define $V(p) = (\uparrow z) \cup \{ \mathbf{\sigma} \}$ and $V(q) = (X \setminus \downarrow z) \cup \{ \mathbf{\sigma} \}$ $\{\mathbf{\sigma}^*\}$. Then $z \Vdash p$ and $z \not\Vdash q$ (because $z \neq \mathbf{\sigma}^*$), and therefore $y \not\Vdash p \to q$, so that $x \not\Vdash \Box(p \to q)$. But this means that we must have $x \not\Vdash \Diamond p \to \Box q$. So there exists some successor v of x such that $v \Vdash \Diamond p$ while $v \nvDash \Box q$. The latter implies that there exist worlds u, w such that $v \leq uRw$ and $w \not\Vdash q$. By definition of V(q) this means $w \leq z$. Furthermore, $v \Vdash \Diamond p$ implies $u \Vdash \Diamond p$, so each intuitionistic successor s of u has a modal successor that satisfies p. By definition this means that either sR or there is some t such that sRt and $z \leq t$. Thus we have found $u, w \in X$ with the desired properties, hence the frame satisfies ($|_{\Diamond \Box}$ -corr). \square



Fig. 3: Correspondence conditions for validity of C_{\diamond} and $I_{\diamond\Box}$. The solid arrows indicate universally quantified arrows, while the dashed ones indicate existential ones. The modal relation is labelled *R* and the intuitionistic relation is unlabelled.

VI. EIGHT COMPLETENESS RESULTS

The correspondence results for the axioms N_{\diamond} , C_{\diamond} and $I_{\diamond \Box}$ give rise to sound semantics for extensions of CK with any combination of them. Next, we prove corresponding completeness results. Where possible, we show that the canonical model satisfies the sufficient frame conditions, to simplify our reasoning about conservativity in Section VII.

Theorem VI.1. Let $Ax \subseteq \{N_{\diamond}, C_{\diamond}, I_{\diamond \square}\}$. Then the logic $CK \oplus Ax$ is sound and strongly complete with respect to the class of frames satisfying (A-suff) for each $A \in Ax$, except for $CK \oplus N_{\diamond}$ and $CK \oplus N_{\diamond} \oplus C_{\diamond}$, for which $(N_{\diamond}$ -corr) replaces $(N_{\diamond}$ -suff).

The completeness part of the proof splits into four cases, depending on whether or not C_{\diamond} and $I_{\diamond\Box}$ are in Ax. Each uses a slightly different canonical model construction. Intuitively, we "prune" the canonical model construction for CK from Definition IV.4 (i.e. we leave out certain segments) to ensure satisfaction of relevant sufficient and correspondence conditions. We begin by adapting the canonical model to accommodate for the case where $C_{\diamond} \in Ax$ and $I_{\diamond\Box} \notin Ax$.

Definition VI.2 (**P**). Let Γ and Δ be prime theories.

- The A-segment of Γ is the segment (Γ, A) where Δ ∈ A if and only if for all φ ∈ L: □φ ∈ Γ implies φ ∈ Δ, and φ ∈ Δ implies ◊φ ∈ Γ.
- 2) The *B*-segment of Γ is the segment (Γ, B) where $\Delta \in B$ if and only if $\{\varphi \mid \Box \varphi \in \Gamma\} \subseteq \Delta$.

Note that we always have $\mathfrak{G} \in B$. Besides, $\mathfrak{G} \in A$ if $\diamond \perp \in \Gamma$. We now verify that (Γ, A) and (Γ, B) are indeed segments.

Lemma VI.3. Let Γ be a prime theory with A- and B-segment (Γ, A) and (Γ, B) . Then we have:

- 1) If $\Box \varphi \in \Gamma$ then $\varphi \in \Delta$ for all $\Delta \in A \cup B$ (\blacksquare).
- If ◊φ ∈ Γ then φ ∈ Δ for some Δ ∈ A, and φ ∈ Θ for some Θ ∈ B (➡).

Proof. Item 1) holds by construction, and for B-segments 2) is witnessed by $\Theta = \mathbf{e}^* \in B$, so we are left to prove 2) for A-segments. To this end, suppose $\Diamond \varphi \in \Gamma$. If $\Diamond \bot \in \Gamma$, then we can also use $\Delta = \mathbf{e}^* \in A$ as witness. Else, we claim that

$$\Box^{-1}(\Gamma) \cup \{\varphi\} \not\vdash \diamondsuit^{-1}(\Gamma^c).$$
(1)

(Here $\Gamma^c = \mathbf{L} \setminus \Gamma$, so $\diamond^{-1}(\Gamma^c) = \{\theta \mid \diamond \theta \notin \Gamma\}$.) Then $\diamond^{-1}(\Gamma^c)$ is not empty as $\diamond \perp \notin \Gamma$. If (1) is false, then there are $\psi_1, \ldots, \psi_n \in \Box^{-1}(\Gamma)$ and $\theta_1, \ldots, \theta_m \in \diamond^{-1}(\Gamma^c)$ such that $\psi_1 \wedge \cdots \wedge \psi_n \wedge \varphi \vdash \theta_1 \vee \cdots \vee \theta_m$. Since \Box distributes over meets and \diamond distributes over joins (\clubsuit), we have $\psi := \psi_1 \wedge \cdots \psi_n \in \Box^{-1}(\Gamma)$ and $\theta := \theta_1 \vee \cdots \vee \theta_m \in \diamond^{-1}(\Gamma^c)$. Then $\psi \wedge \varphi \vdash \theta$, so by $\mathsf{K}_{\diamond}, \ \Box \psi \wedge \diamond \varphi \vdash \diamond \theta$. By assumption $\Box \psi, \diamond \varphi \in \Gamma$, hence $\diamond \theta \in \Gamma$, a contradiction. Now the Lindenbaum lemma yields a prime theory Δ containing $\Box^{-1}(\Gamma) \cup \{\varphi\}$ and disjoint from $\diamond^{-1}(\Gamma^c)$. Therefore $\varphi \in \Delta$ and $\Delta \in A$, as desired.

We construct the canonical frame \mathfrak{X}_{AB} and model \mathfrak{M}_{AB} (\triangleright) as in Definition IV.4, except that we restrict our worlds to A- and B-segments. Then we have the following truth lemma.

Lemma VI.4 (Truth lemma **P**). Let (Γ, U) be an A- or Bsegment. Then for all $\varphi \in \mathbf{L}$ we have $(\Gamma, U) \Vdash \varphi$ iff $\varphi \in \Gamma$.

Proof. By induction on the structure of φ . We demonstrate the case $\varphi = \Box \psi$. If $\Box \psi \in \Gamma$ then $\Gamma \Vdash \Box \psi$ by construction. If $\Box \psi \notin \Gamma$ then $\Box^{-1}(\Gamma) \nvDash \psi$, so we can use the Lindenbaum lemma to find a prime theory Δ containing $\Box^{-1}(\Gamma)$ but not ψ . By definition Δ is in the tail of the B-segment of Γ . Extending Δ to a segment (Δ, V) (using either the A- or B-segment), we find $(\Gamma, U) \lesssim (\Gamma, B)R(\Delta, V)$, and by the induction hypothesis $(\Delta, V) \nvDash \psi$. Therefore $(\Gamma, U) \nvDash \Box \psi$. \Box

Finally we check that restricting to A- and B-segments gives rise to a frame that satisfies (C_{\diamond} -suff).

Lemma VI.5 (**P**). *The frame* \mathfrak{X}_{AB} *satisfies* (C_{\Diamond} -suff).

Proof. Given (Γ, U) , we claim that the A-segment (Γ, A) of Γ witnesses satisfaction of $(\mathbb{C}_{\diamond}$ -suff). Suppose $(\Gamma, U) \subseteq (\Gamma', U')$ and $(\Gamma, A)R(\Delta, V)$ (so $\Delta \in A$). If $\mathfrak{C} \in U'$ then (\mathfrak{C}, A) is a segment such that $(\Gamma', U')R(\mathfrak{C}, A)$ and $(\Delta, V) \subseteq (\mathfrak{C}, A)$, so $(\mathbb{C}_{\diamond}$ -suff) holds. So assume $\mathfrak{C} \notin U'$. Then (Γ', U') is an A-segment. We construct a $\Delta' \in U'$ such that $\Delta \subseteq \Delta'$. Then e.g. the A-segment (Δ', A) witnesses truth of $(\mathbb{C}_{\diamond}$ -suff).

First, note that $\{\psi \mid \Diamond \psi \notin \Gamma'\} \neq \emptyset$, else $\Diamond \perp \in \Gamma'$ would entail the presence of **e** in U'. Second, we claim that

$$\Box^{-1}(\Gamma') \cup \Delta \not\vdash \{\psi \mid \Diamond \psi \notin \Gamma'\}.$$
⁽²⁾

If this were not the case, then we can find $\Box \varphi \in \Gamma'$ and $\delta \in \Delta$ and $\Diamond \psi \notin \Gamma'$ such that $\varphi \land \delta \vdash \psi$, hence $\Box \varphi \land \Diamond \delta \vdash \Diamond \psi$. By definition of A-segments, $\delta \in \Delta$ implies $\Diamond \delta \in \Gamma$, so $\Diamond \delta \in \Gamma'$. But $\Box \varphi \in \Gamma'$ by construction, hence $\Diamond \psi \in \Gamma'$, a contradiction. So (2) holds, and we can use the Lindenbaum lemma to find a prime theory Δ' containing Δ and φ for each $\Box \varphi \in \Gamma'$, while avoiding ψ for each $\Diamond \psi \notin \Gamma'$. We claim that $\Diamond \varphi \in \Gamma'$ for each $\varphi \in \Delta'$. If not, then there exists a $\varphi \in \Delta'$ such that $\Diamond \varphi \notin \Gamma'$, contradicting the fact that Δ' avoids all ψ such that $\Diamond \psi \notin \Gamma'$. This entails $\Delta \subseteq \Delta'$ and $\Delta' \in U'$.

Combining the previous lemmas yields:

Lemma VI.6 (P). $CK \oplus C_{\diamond}$ is sound and strongly complete with respect to the class of frames satisfying (C_{\diamond} -suff).

Next, we consider the case where $I_{\Diamond \Box} \in Ax$ and $C_{\Diamond} \notin Ax$.

Definition VI.7 (**P**). A *P*-segment (or purposeful segment) of Γ is defined as follows:

- If ⊥ ∈ Γ, then the only P-segment based on Γ is (Γ, {Γ}) (i.e. (𝔅, {𝔅}));
- If $\perp \notin \Gamma$ but $\diamond \perp \in \Gamma$, then the only P-segment based on Γ is (Γ, U) , where $U = \{\Delta \mid \Box^{-1}(\Gamma) \subseteq \Delta\}$;
- If ⊥, ◇⊥ ∉ Γ, then a segment (Γ, U) is a P-segment if there exists a ◇π ∉ Γ such that

$$U = \{ \Delta \mid \Box^{-1}(\Gamma) \subseteq \Delta \text{ and } \pi \notin \Delta \} =: U_{\Gamma, \pi}.$$

It is clear that the two cases with $\Diamond \perp \in \Gamma$ give rise to segments. For the case with $\Diamond \perp \notin \Gamma$, we have:

Lemma VI.8 (**p**). Let Γ be a prime theory such that $\Diamond \pi \notin \Gamma$. Let $U_{\Gamma,\pi} := \{\Delta \mid \Box^{-1}(\Gamma) \subseteq \Delta \text{ and } \pi \notin \Delta\}$. Then $(\Gamma, U_{\Gamma,\pi})$ is a segment.

Proof. By definition $\Box \varphi \in \Gamma$ implies $\varphi \in \Delta$. Let $\Diamond \varphi \in \Gamma$. Then $\Box^{-1}(\Gamma) \cup \{\varphi\} \not\vDash \pi$, for if this were not the case then $\Gamma, \Diamond \varphi \vdash \Diamond \pi$, contradicting $\Diamond \pi \notin \Gamma$. So the Lindenbaum lemma yields a prime theory Δ containing $\Box^{-1}(\Gamma)$ and φ but not π , whence $\Delta \in U$ and $\varphi \in \Delta$ as desired. \Box

As a consequence of this lemma, for any prime theory Γ there exists a P-segment headed by Γ . Construct the canonical frame \mathfrak{X}_P and model \mathfrak{M}_P as in Definition IV.4, but restricting the set of segments to P-segments (\mathbb{P}).

Lemma VI.9 (Truth lemma **P**). Let (Γ, U) be a *P*-segment in \mathfrak{M}_P . Then for all $\varphi \in \mathbf{L}$ we have $(\Gamma, U) \Vdash \varphi$ iff $\varphi \in \Gamma$.

Proof. By induction on the structure of φ . The only interesting cases are for $\varphi = \Box \psi$ and $\varphi = \Diamond \psi$.

If $\Box \psi \in \Gamma$ then $\Gamma \Vdash \Box \psi$ by construction. If $\Box \psi \notin \Gamma$ then $\Box^{-1}(\Gamma) \nvDash \psi$. Use the Lindenbaum lemma to construct a prime theory Δ containing $\Box^{-1}(\Gamma)$ but not ψ , and extend this to a P-segment (Δ, V) . Now we consider two cases: if $\Diamond \bot \in \Gamma$ then Δ is in the tail of the (unique) P-segment headed by Γ (by definition). Then $(\Gamma, U)R(\Delta, V)$ and $(\Delta, V) \nvDash \psi$, hence $(\Gamma, U) \nvDash \Box \psi$. If $\Diamond \bot \notin \Gamma$ then $\Delta \in U_{\Gamma, \bot}$, so $(\Gamma, U_{\Gamma, \bot})$ is a segment above (Γ, U) witnessing $(\Gamma, U) \nvDash \Box \psi$.

If $\diamond \psi \in \Gamma$ then $(\Gamma, U) \Vdash \diamond \psi$ by the definition of segments. If $\diamond \psi \notin \Gamma$ then Lemma VI.8 yields a segment (Γ, V) such that no $\Delta \in V$ contains ψ . By induction we then find $(\Gamma, V) \not\Vdash \diamond \psi$, and since $(\Gamma, U) \subseteq (\Gamma, V)$ this implies $(\Gamma, U) \not\Vdash \diamond \psi$. \Box

We show that the canonical model satisfies ($I_{\Diamond \Box}$ -suff).

Lemma VI.10 (**P**). \mathfrak{X}_P satisfies ($|_{\Diamond \square}$ -suff).

Proof. Suppose $(\Gamma, U)R(\Delta, V) \subseteq (\Delta', V')$. If $(\Delta', V') = (\textcircled{O}, \{\textcircled{O}\})$ then we can take $u = (\textcircled{O}, \{\textcircled{O}\})$ to satisfy $(I_{\Diamond \Box}$ -suff). So suppose this is not the case. Then $\bot \notin \Delta'$. We claim that

$$\Gamma \cup \Diamond(\Delta') \not\vdash \Box((\delta')^c). \tag{3}$$

If this were false, then using the fact that Γ and Δ' are prime we can find $\varphi \in \Gamma$, $\delta \in \Delta'$ and $\psi \notin \Delta'$ such that $\varphi \land \Diamond \delta \vdash \Box \psi$. Then $\varphi \vdash \Diamond \delta \rightarrow \Box \psi$, so $\varphi \vdash \Box (\delta \rightarrow \psi)$ by $I_{\Diamond \Box}$, hence $\Box (\delta \rightarrow \psi) \in \Gamma$, wherefore $\delta \rightarrow \psi \in \Delta \subseteq \Delta'$. By assumption $\delta \in \Delta'$, so deductive closure of Δ' implies $\psi \in \Delta'$, a contradiction.

So (3) holds, and the Lindenbaum lemma yields a prime theory Θ that contains Γ and $\diamond(\Delta')$ but avoids $\Box((\Delta')^c)$. If $\diamond \bot \in \Theta$ then Θ is the head of a unique P-segment (Θ, U') , and $(\Gamma, U) \subseteq (\Theta, U')R(\Delta', V')$ by construction. Moreover, if $(\Theta, U') \subseteq (\Gamma'', U'')$ then $\diamond \bot \in \Gamma''$, hence $\mathfrak{C} \in U''$ and $(\mathfrak{C}, \{\mathfrak{C}\})$ is a world such that $(\Gamma'', U'')R(\mathfrak{C}, \{\mathfrak{C}\})$ and $(\Delta', V') \subseteq (\mathfrak{C}, \{\mathfrak{C}\})$. So $(I_{\diamond \Box}$ -suff) is satisfied.

If $\diamond \perp \notin \Theta$, then we take $(\Theta, U_{\Theta,\perp})$. By construction $(\Gamma, U) \subseteq (\Theta, U_{\Theta,\perp}) R(\Delta', V')$. Let (Γ'', U'') be a P-segment such that $\Theta \subseteq \Gamma''$. If $\perp \in \gamma''$ or $\diamond \perp \in \Gamma''$ then $\mathbf{e} \in U''$ and

we can use $(\mathbf{G}^{*}, \{\mathbf{G}^{*}\})$ to witness $(\mathsf{I}_{\Diamond\square}\text{-suff})$. If not, then U'' must be of the form $U_{\Gamma'',\pi}$ for some $\Diamond \pi \notin \Gamma''$. We claim that

$$\Box^{-1}(\Gamma'') \cup \Delta' \not\vdash \pi. \tag{4}$$

If this were false, then there exist $\varphi \in \Box^{-1}(\Gamma'')$ and $\delta \in \Delta'$ such that $\varphi \land \delta \vdash \pi$. This implies $\Box \varphi \land \Diamond \delta \vdash \Diamond \pi$, and since $\Box \varphi, \Diamond \delta \in \Gamma''$ we get $\Diamond \pi \in \Gamma''$, a contradiction. Using the Lindenbaum lemma, (4) yields a prime theory Δ'' that we can extend to a segment (Δ'', V'') . By construction $\Delta'' \in U''$, so $(\Gamma'', U'')R(\Delta'', V'')$, as well as $\Delta' \subseteq \Delta''$.

Lemma VI.11 (P). $CK \oplus I_{\Diamond \Box}$ is sound and strongly complete with respect to the class of frames satisfying $(I_{\Diamond \Box} \text{-corr})$.

We have now gathered enough background to prove:

Proof of Theorem VI.1. Soundness results are straightforwardly obtained as instances of a general result (**)** via the use of correspondence conditions (**)**, **)**, **)**, **)**, **)**, **)**, For completeness, we consider four cases based on whether or not C_{\diamond} and $I_{\diamond\square}$ are in Ax. In each case, adding N_{\diamond} to the logic yields a notion of (prime) theory which implies that the corresponding canonical model satisfies (N_{\diamond} -corr) or (N_{\diamond} -suff).

Case 1: $C_{\diamond} \notin Ax$ *and* $I_{\diamond \Box} \notin Ax$. If $N_{\diamond} \notin Ax$ then this is Theorem IV.9 (P) and, as mentioned, if $N_{\diamond} \in Ax$ then the canonical model construction used in this theorem satisfies $(N_{\diamond}$ -corr) (P).

Case 2: $C_{\diamond} \in Ax$ and $I_{\diamond \Box} \notin Ax$. Use a canonical model based on A- and B-segments as outlined above ($\blacksquare, \blacksquare$).

Case 3: $C_{\diamond} \notin Ax$ and $I_{\diamond \Box} \in Ax$. In this case we use a canonical model based on P-segments, as outlined above (**P**, **P**).

Case 4: $C_{\diamond} \in Ax$ and $I_{\diamond\square} \in Ax$. In this final case we can use prime theories as the worlds of our canonical models (A), rather than segments. These are partially ordered by inclusion, and we define the modal accessibility relation R by letting $\Gamma R\Delta$ iff $\Box^{-1}(\Gamma) \subseteq \Delta$ and $\diamond(\Delta) \subseteq \Gamma$. We can view this as the restriction to segments of the form (Γ, U) where a prime theory Δ is in U if and only if $\Gamma R\Delta$. It turns out that C_{\diamond} is crucial to prove the truth lemma, while $I_{\diamond\Box}$ allows us to prove that the resulting frame satisfies (C_{\diamond} -strong) and ($I_{\diamond\Box}$ -weak) (A). Once again, we obtain strong completeness (A).

Completeness with respect to correspondence conditions straightforwardly follows.

Corollary VI.12. Let $Ax \subseteq \{N_{\diamond}, C_{\diamond}, I_{\diamond\Box}\}$. Then the logic $CK \oplus Ax$ is sound and strongly complete with respect to the class of frames satisfying (A-corr) for each $A \in Ax$.

Proof. Note that the cases where $Ax \subseteq \{N_{\diamond}\}$ are already treated above. For the remaining ones, as sufficient conditions entail correspondence conditions $(\mathbb{P},\mathbb{P},\mathbb{P},\mathbb{P})$, we directly use Theorem VI.1 to obtain our result $(\mathbb{P},\mathbb{P},\mathbb{P},\mathbb{P},\mathbb{P},\mathbb{P},\mathbb{P})$.

VII. COMPARISON OF DIAMOND-FREE FRAGMENTS

We leverage the sound and complete semantics for extensions of CK to study their diamond-free fragments. We call a logic a *conservative extension of* CK_{\Box} if its \diamond -free fragment coincides with CK_{\Box} . It is known that the \diamond -free fragment of $CK \oplus N_{\diamond} \oplus C_{\diamond}$ is a conservative extension of CK_{\Box} [19, Corollary 30], and that this is not the case for $CK \oplus N_{\diamond} \oplus I_{\diamond \Box}$. In a blog comment [17] Das speculated that "the real distinction of IK is due to $[I_{\diamond \Box}]$ ", but we falsify this by proving that $CK \oplus C_{\diamond} \oplus I_{\diamond \Box}$ (hence $CK \oplus I_{\diamond \Box}$) is conservative over CK_{\Box} . As a consequence, we characterise precisely which extensions of CK with axioms in $\{N_{\diamond}, C_{\diamond}, I_{\diamond \Box}\}$ are conservative over CK_{\Box} .

Proposition VII.1 (A). The logic $CK \oplus C_{\diamond} \oplus I_{\diamond \Box}$ is a conservative extension of CK_{\Box} .

Proof. By definition every formula that is derivable in CK_{\Box} is also derivable in $CK \oplus C_{\Diamond} \oplus I_{\Diamond \Box}$, so we focus on the converse. We show that for every world in every CK-frame we can find a world in a frame for $CK \oplus C_{\Diamond} \oplus I_{\Diamond \Box}$ that satisfies exactly the same diamond-free formulas. This implies that the class of CK-frames validates all diamond-free consecutions in $CK \oplus C_{\Diamond} \oplus I_{\Diamond \Box}$, so that the result follows from completeness of CK_{\Box} with respect to the class of CK-frames.

Let $\mathfrak{X} = (X, \mathfrak{G}, \leq, R)$ be a CK-frame, and define binary relations \leq and \mathcal{R} on X by:

$$x \leq y$$
 iff $x \leq y$ or $y = \textcircled{a}^*$
 $x \mathcal{R} y$ iff $x \mathcal{R} y$ or $y = \textcircled{a}^*$

(Note that \mathfrak{G} was already maximal with respect to \leq , but now it becomes a top element with respect to \leq .) Then $\mathfrak{X}' = (X, \mathfrak{G}, \leq, \mathcal{R})$ is a CK-frame (\clubsuit). The fact that every world can intuitionistically and modally access \mathfrak{G} implies that \mathfrak{X}' satisfies (C_{\diamond} -strong) (\clubsuit). Moreover, we have that $x\mathcal{R}y \leq z$ implies $x \leq x\mathcal{R}y \leq z$ and all intuitionistic successors of x can modally access \mathfrak{G} , wherefore ($\mathsf{I}_{\diamond \square}$ -corr) holds (\clubsuit). Therefore \mathfrak{X}' is a frame for $\mathsf{CK} \oplus \mathsf{C}_{\diamond} \oplus \mathsf{I}_{\diamond \square}$.

Valuations for \mathfrak{X} and \mathfrak{X}' coincide because we only added relations of the form $x \leq \mathfrak{G}$ to the intuitionistic accessibility relation. Let V be such a valuation for \mathfrak{X} . Then we can show by induction on the structure of φ that

$$(\mathfrak{X}, V), x \Vdash \varphi \quad \text{iff} \quad (\mathfrak{X}', V), x \Vdash \varphi$$

for all diamond-free formulas φ (\clubsuit). This entails that every diamond-free consecution $\Gamma \vdash \varphi$ in $\mathsf{CK} \oplus \mathsf{C}_{\Diamond} \oplus \mathsf{I}_{\Diamond\square}$ is true at x. Since \mathfrak{X} , V and x are arbitrary, the result follows.

We can also give a semantical proof for $CK \oplus N_{\diamond} \oplus C_{\diamond}$ [19, Corollary 30]. This uses a different frame transformation: whereas the proof of Proposition VII.1 makes o modally accessible to all worlds to force validity of $\diamond \perp$, the frame transformation in the next proposition falsifies $\diamond \top$ in all

⁵Formally, we show that $CK \oplus Ax$ and CK coincide on their \diamond -free fragments for certain Ax, relying on Das and Marin [19, Corollary 6] that this coincides with CK_{\Box} .

worlds.⁶ Both approaches ensure validity of C_{\diamond} by trivialising diamonds.

Proposition VII.2 (A). The logic $CK \oplus C_{\diamond} \oplus N_{\diamond}$ is a conservative extension of CK_{\Box} .

Proof. We use the same strategy as in the proof of Proposition VII.1. Let $\mathfrak{X} = (X, \mathfrak{C}, \leq, R)$ be any CK-frame. We construct a new frame \mathfrak{X}' by adding for each world $x \neq \mathfrak{C}$ an additional world x^+ that satisfies $x \leq x^+ \leq x$ and that cannot modally see anything. Intuitively, this makes $\diamond \top$ false at every world (except \mathfrak{C}), forcing distributivity of diamonds over joins.

Let X^+ be a copy of $X \setminus \{ \mathfrak{G} \}$, and for each $\mathfrak{G} \neq x \in X$ denote its copy in X^+ by x^+ .⁷ Define a preorder \leq on $X \cup X^+$ by

$$\preceq := \bigcup \left\{ \{ (x, y), (x, y^+), (x^+, y), (x^+, y^+) \} \\ | x, y \in X \text{ and } x \le y \right\}.$$

Leave R unchanged, but view it as a relation on $X \cup X^+$. Then $\mathfrak{X}' = (X \cup X^+, \mathfrak{o}, \preceq, R)$ is a CK-frame (A). The worlds x^+ ensure satisfaction of $(\mathbb{N}_{\diamond}$ -corr) (A). Furthermore, $(\mathbb{C}_{\diamond}$ -corr) is satisfied (A), because if $x \preceq y$ and $x \preceq z$ then $x^+ \succeq x$ is a world such that $R[x^+] \subseteq \downarrow R[y]$ and $R[x^+] \subseteq \downarrow R[z]$ (since $R[x^+] = \emptyset$). So \mathfrak{X}' is a frame for $\mathsf{CK} \oplus \mathsf{C}_{\diamond} \oplus \mathsf{N}_{\diamond}$.

Let V be any valuation for \mathfrak{X} and define a valuation V' for \mathfrak{X}' by $V'(p) = V(p) \cup \{x^+ \mid x \in V(p)\}$. Then a routine induction on the structure of φ yields $(\mathfrak{X}, V), x \Vdash \varphi$ iff $(\mathfrak{X}', V'), x \Vdash \varphi$ for every diamond-free formula φ (**P**). The remainder of the proof is as in Proposition VII.1.

Summarising our results, we get:

Theorem VII.3. Let $Ax \subseteq \{N_{\diamond}, C_{\diamond}, I_{\diamond \Box}\}$. Then $CK \oplus Ax$ is a conservative extension of CK_{\Box} if and only if $N_{\diamond} \notin Ax$ or $I_{\diamond \Box} \notin Ax$.

Proof. Non-conservativity of $\{N_{\diamond}, I_{\diamond\Box}\}$ (\clubsuit) and $\{N_{\diamond}, C_{\diamond}, I_{\diamond\Box}\}$ (\clubsuit) is witnessed by the fact that they prove $\neg \neg \Box \bot \rightarrow \Box \bot$ [19], while CK does not. Conservativity of the remaining extensions ($\clubsuit, \clubsuit, \clubsuit, \clubsuit$) follows from Propositions VII.1 and VII.2.

Theorem VII.3 leaves us with two logics that are not conservative over CK_{\Box} . While clearly $CK \oplus N_{\Diamond} \oplus I_{\Diamond \Box}$ is included in IK, it is unclear whether or not their diamond-free fragments coincide.

Open question VII.4. Does the \diamond -free fragment of $CK \oplus N_{\diamond} \oplus I_{\diamond \Box}$ coincide with that of $CK \oplus N_{\diamond} \oplus I_{\diamond \Box} \oplus C_{\diamond}$? Are either or both fragments finitely axiomatisable?

VIII. OTHER AXIOMS

In Sections V, VI and VII we focussed on logics between CK and IK generated by three axioms. There are of course myriad axioms to be considered, which can all be investigated semantically. As an example of this, we study two more axioms that have appeared in the literature.

A. The weak normality axiom

We first consider the *weak normality axiom*:

 $(N_{\Diamond\Box}) \quad \Diamond \bot \to \Box \bot$

This axiom was used by Kojima [41] in the context of neighbourhood semantics, weakening WK by replacing N_{\Diamond} with $N_{\Diamond\square}$. We investigate how this axiom fits in our semantic framework, and use it to compare extensions of CK that include $N_{\Diamond\square}$. This gives rise to logics whose diamond-free fragments do not coincide with the ones from Section VII.

Proposition VIII.1. A frame validates $N_{\Diamond\square}$ whenever it satisfies (\clubsuit):

 $(N_{\Diamond \Box}$ -suff) $\forall x \ (if \ xR \bullet and \ xRy \ then \ y = \bullet)$

Moreover, a frame validates $\mathbb{N}_{\diamond\square}$ if and only if it satisfies (\clubsuit): $(\mathbb{N}_{\diamond\square}\text{-corr}) \quad \forall x (if \forall y \ (x \leq y \text{ implies } yR))$ then $\forall y, z \ (x \leq yRz \text{ implies } z = \bullet)$

Proof. The first claim follows from the second, because $(N_{\diamond \Box}$ -suff) implies $(N_{\diamond \Box}$ -corr) (\clubsuit). The second claim follows from a routine verification.

What happens if we add $N_{\Diamond\square}$ to any of the eight logical systems discussed in Section V? Since N_{\Diamond} implies $N_{\Diamond\square}$, adding it to a system containing N_{\Diamond} does not change anything. Furthermore, Theorem VII.3 implies that Kojima's logic $CK \oplus N_{\Diamond\square}$, as well as its extension with C_{\Diamond} , are conservative over CK_{\Box} ($\blacksquare,\blacksquare$). This leaves us with two new logics (in terms of their diamond-free fragment) between CK_{\Box} and IK. We claim that they relate as in Figure 4.

Theorem VIII.2. Let $Ax \subseteq \{C_{\diamond}, I_{\diamond\Box}\}$. Then the logic $CK \oplus N_{\diamond\Box} \oplus Ax$ is sound and strongly complete with respect to the class of frames satisfying $(N_{\diamond\Box}\text{-corr})$, and (A-suff) for each $A \in Ax$ (\clubsuit , \clubsuit). Moreover, if $I_{\diamond\Box} \in Ax$ then we can replace $(N_{\diamond\Box}\text{-corr})$ by $(N_{\diamond\Box}\text{-suff})$ (\clubsuit , \clubsuit).

The next propositions witness the non-inclusions indicated in Figure 4.

Proposition VIII.3 (P). The formula $\neg \neg \Box p \rightarrow \Box \neg \neg p$ is derivable in $CK \oplus N_{\diamond} \oplus I_{\diamond \Box}$ but not in $CK \oplus N_{\diamond \Box} \oplus C_{\diamond} \oplus I_{\diamond \Box}$ (hence not in $CK \oplus N_{\diamond \Box} \oplus I_{\diamond \Box}$).

Proof. Derivability in $CK \oplus N_{\Diamond} \oplus I_{\Diamond\square}$ is a known result [19, Lemma 10]. Figure 5a gives a frame that satisfies ($N_{\Diamond\square}$ -suff),

⁶Our frame transformation for $\mathsf{CK} \oplus \mathsf{C}_{\diamond} \oplus \mathsf{I}_{\diamond\square}$ tightly connects to the translation given in an independent alternative proof of Proposition VII.1 [43] which maps all diamond formulas to \bot . Following this insight, the result for $\mathsf{CK} \oplus \mathsf{N}_{\diamond} \oplus \mathsf{C}_{\diamond}$ can be obtained by translating diamond formulas to \top instead [18].

⁷In the formalisation, X^+ is taken to include a copy of \checkmark . This copy is forced to be isolated as it is only intuitionistically or modally accessible from itself. Therefore it does not affect validity in the corresponding model.



Fig. 4: Logics including $N_{\Diamond\square}$. Inclusion of logics from left to right is immediate. Dashed arrows indicate non-inclusion of diamond-free fragments, and follow from Propositions VIII.3 and VIII.4. We note that non-inclusion of the middle question mark would imply non-inclusion of the other two.

 $(C_{\diamond}$ -strong) and $(I_{\diamond\Box}$ -suff) where the formula is not valid (it is false at x if we take a valuation with $V(p) = \{ \mathbf{\sigma}^{*} \}$), hence it is not derivable in $CK \oplus N_{\diamond\Box} \oplus I_{\diamond\Box}$.

Proposition VIII.4 (P). *The formula*

$$\neg\Box\bot \to (\neg\neg\Box p \to \Box\neg\neg p). \tag{5}$$

is derivable in $CK \oplus N_{\Diamond \Box} \oplus I_{\Diamond \Box}$, but not in CK.

Proof. To see that the formula is not derivable in CK, consider the frame from Figure 5b with valuation $V(p) = \{w, \mathbf{e}\}$. Then clearly x and y do not satisfy $\Box \bot$, hence $x \Vdash \neg \Box \bot$. Furthermore, the fact that $y \Vdash \Box p$ implies that neither x nor y satisfies $\neg \Box p$, and hence $x \Vdash \neg \neg \Box p$. Finally, $z \nvDash \neg \neg p$ because $z \Vdash \neg p$, which implies $x \nvDash \Box \neg \neg p$. Combining this entails that x falsifies the formula.

Next, we derive the formula in $CK \oplus N_{\Diamond\square} \oplus I_{\Diamond\square}$. Applying Nec, K_{\Box} and MP to $p \to \neg \neg p$, a theorem of IPL, yields $\square p \to$ $\square \neg \neg p$. This is equal to $\square p \to \square(\neg p \to \bot)$, so K_{\Diamond} yields $\square p \to (\Diamond \neg p \to \Diamond \bot)$. Using $N_{\Diamond\square}$ gives $\square p \to (\Diamond \neg p \to \square \bot)$. Using propositional intuitionistic reasoning we rewrite this to

$$\neg\neg\Box p \to (\neg\Box\bot \to \neg\Diamond\neg p).$$

Now currying and commutativity of \land allows us to derive $\neg\Box \bot \rightarrow (\neg\neg\Box p \rightarrow \neg \Diamond \neg p)$. Rewriting $\neg \Diamond \neg p$ to $\Diamond \neg p \rightarrow \bot$ and using the fact that $\bot \rightarrow \Box \bot$ yields

$$\neg\Box\bot \to (\neg\neg\Box p \to (\Diamond\neg p \to \Box\bot)).$$

Finally, applying $I_{\Diamond \Box}$ we find $\neg \Box \bot \rightarrow (\neg \neg \Box p \rightarrow \Box \neg \neg p)$. \Box



(a) Frame for Prop. VIII.3. (b) Frame for Proposition VIII.8.

Fig. 5: Two more frames used to falsify formulas.

We have the following analogue of Open question VII.4.

Open question VIII.5. Does the \diamond -free fragment of CK \oplus $N_{\diamond\square} \oplus I_{\diamond\square}$ coincide with that of CK \oplus $N_{\diamond\square} \oplus I_{\diamond\square} \oplus C_{\diamond}$? Are the \diamond -free fragments of either of these logics finitely axiomatisable?

B. The weak constant domain axiom

The logic FIK arises from IK by replacing $I_{\Diamond\Box}$ with the *weak constant domain* axiom:

$$(\mathsf{wCD}) \quad \Box(p \lor q) \to ((\diamondsuit p \to \Box q) \to \Box q)$$

It was introduced by Balbiani, Gao, Gencer, and Olivetti [4] to axiomatise the intuitionistic modal logic with birelational semantics satisfying (C_{\Diamond} -strong), but not necessarily ($I_{\Diamond\square}$ -weak). We briefly investigate wCD and how it relates to the axioms from Section V to obtain results similar to those of Figure 4. We postpone completeness results and finite axiomatisability of some of the diamond-free fragments to future work.

Proposition VIII.6 (**P**). Over CK, the axiom $I_{\Diamond \Box}$ implies wCD.

Proof. Suppose we have $(\Diamond p \to \Box q) \to \Box (p \to q)$. Then it follows that

$$\Box(p \lor q) \to ((\diamondsuit p \to \Box q) \to (\Box(p \lor q) \land \Box(p \to q))).$$

The conjunction is equal to $\Box((p \lor q) \land (p \to q))$, and since $((p \lor q) \land (p \to q)) \to q$ is a theorem of intuitionistic logic it implies $\Box q$. Thus we find $\Box(p \lor q) \to ((\diamondsuit p \to \Box q) \to \Box q$. \Box

The previous proposition implies that adding wCD to a logic that already proves $I_{\Diamond\Box}$ does not add any extra deductive power. So adding wCD to the logics from Section V yields four logics of interest: $CK \oplus wCD$, $CK \oplus C_{\Diamond} \oplus wCD$, $CK \oplus N_{\Diamond} \oplus wCD$, and FIK. The diamond-free fragments of the first two coincide with CK_{\Box} as a consequence of Theorem VII.3. Thus we are left with a situation similar to Figure 4, depicted in Figure 6.

Proposition VIII.7. *The formula* $\neg \neg \Box p \rightarrow \Box \neg \neg p$ *is derivable in* CK $\oplus \mathbb{N}_{\diamond} \oplus \mathbb{I}_{\diamond \Box}$ (**P**) *but not in* FIK.

Proof. The first part of the statement is a known result [19, Lemma 10]. The second part follows from constructing a countermodel in the semantics for FIK [4]. \Box

Proposition VIII.8 (P). The formula

$$(\mathsf{wCD}_{\Box}) \quad \Box(p \lor q) \to ((\neg \Box \neg p \to \Box q) \to \Box q)$$

is derivable in $\mathsf{CK} \oplus \mathsf{N}_{\Diamond} \oplus \mathsf{wCD}$ but not in $\mathsf{CK}.$

Proof. The formula $\Diamond p \to \neg \Box \neg p$ is a theorem of $\mathsf{CK} \oplus \mathsf{N}_{\Diamond}$. Now using instances of the intuitionistic theorems

$$\begin{split} (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi)) \\ \text{and} \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \end{split}$$



Fig. 6: Logics including wCD. Inclusion of logics from left to right is immediate. Dashed arrows indicate non-inclusion of diamond-free fragments, and follow from Propositions VIII.7 and VIII.8.

we can derive wCD_{\Box} from wCD. The fact that wCD_{\Box} is not derivable in CK can be shown by taking the CK-frame from Figure 5b and setting $V(p) = \{z, \mathbf{\sigma}^*\}$ and $V(q) = \{w, \mathbf{\sigma}^*\}$. The resulting model falsifies wCD_{\Box} at x.

Open question VIII.9. Does the \diamond -free fragment of $CK \oplus N_{\diamond} \oplus wCD$ coincide with that of $CK \oplus N_{\diamond} \oplus wCD \oplus C_{\diamond}$? Are the \diamond -free fragments of either of these logics finitely axiomatisable?

IX. A NOTE ON THE FORMALISATION

Beyond the extra confidence it provides in our results, for example clarifying the semantics of WK which have been disputed [50], our use of formalisation in the Rocq Prover has been crucial for our work.

Because of the multitude of axiomatic extensions of CK we consider, we had to formalise most of our results parametrised by an arbitrary set of axioms. This generality led to formalised results which are both insightful and close to their pen-and-paper counterparts by their size and structure. For example, while it is often questioned in modal logic [35], we proved the deduction theorem for *all* axiomatic extensions of CK. Additionally, each of our strong completeness results via a canonical model construction, roughly 500 lines of code, was leveraged for several logics, around 50 lines only per logic.

Pragmatically, the code for a certain set of logics could be copied, pasted, and modified for another such set in a heuristic and efficient way. This enabled us to study a total of 12 logics, with an additional few days of work for each further axiom analysed. We invite our readers to experience it for themselves by downloading our code and adding their favourite axioms.

X. FURTHER WORK

We have shown that using a relational semantics for CK, and employing Rocq to verify our proofs and tame the profusion of logics arising from combinations of axioms, allows us to analyse the modal logics between CK and IK. The success of this methodology leaves us with much further work to pursue.

a) Open questions: While our work has closed some open questions about the logics between CK and IK, others remain. Most pressingly, do the \diamond -free fragments of $CK \oplus N_{\diamond} \oplus I_{\diamond \Box}$ and IK coincide, and do they have a finite axiomatisation? We are aware of a discussion, reported by Das and Marin [19], that the fragment of IK might not have a finite axiomatisation, but we are not aware of any proof of this; Grefe's thesis [34] proves only the much weaker result that there exists an infinite chain of finitely axiomatisable \diamond free logics between CK and IK. b) Finite model properties: As noted in Remark IV.8, the canonical model construction for CK can be carried out relative to a set Σ of formulas. This yields the finite model property for CK, as proved already [50, Section 4]. Since the completeness results in Theorem IV.7 use various model constructions, the finite model property does not readily carry over. This raises the question whether we can obtain finite model properties for the logics between CK and IK.

c) Axiomatic extensions: Focus on the K family of logics is reasonable to answer basic questions in modal logic, but much interest concerns extensions of a basic modal logic with further axioms, from S4 to provability logic to epistemic logic. How does the choice of base logic, from CK to IK, effect the properties of these logics? Adding even more axioms into the mix will make our Rocq framework more crucial than ever.

d) Weaker or incomparable basic logics: While a lot of the work in intuitionistic modal logic builds on at least CK, this is not universal. For example, Božić and Došen [9], Wolter and Zakharyaschev [74], [75], and Goré, Postniece and Tiu [33] take as basis necessitation, K_{\Box} , N_{\diamond}, and C_{\diamond}, which is incomparable with CK because K_{\diamond} is merely an optional extension. This setup, with no assumed links between \Box and \diamond , is too weak for a sensible birelational semantics, so instead takes a *tri*relational semantics with separate relations for the two modalities. However if we wished to remain within birelational semantics we could still consider logics where \Box and \diamond interact, but not via K_{\diamond}, or logics with weaker properties of \Box , forgoing necessitation or K_{\Box}.

e) Sahlqvist theorems: Finding a link between axioms and relational properties can take ingenuity. Sahlqvist theorems for classical modal logic [30], [63], [64] make this automatic for certain syntactically defined classes of formulas. If these techniques could be modified to take the relational semantics for CK as their basis, they would provide another tool for taming the jungle of logics in this space.

f) Proof theory and types: There is a body of research too large to summarise on proof theory for intuitionistic modal logics, but our work may help to identify new logics to target and techniques to use, such as labelled sequents that take CK-semantics as their basis. In particular, much recent work on proofs for intuitionistic modal logic involves calculi with type assignment, but no \diamond . These type theories support impressive applications in formalised mathematics, so it may be worthwhile to investigate type theory with \diamond from a CK base, following Bellin, De Paiva and Ritter [5], while simultaneously taking cues from more recent work on modal type theory.

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