Irreducible Objects and Lattice Homomorphisms in Adhesive Categories

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Abstract. It is well-known that the set of subobjects of an object in an adhesive category forms a distributive lattice. This is a work-in-progress paper where we review the lattice-theoretic representation theorem for finite distributive lattices and show how it applies to subobject lattices. Furthermore we show that every arrow in an adhesive category can be interpreted as a lattice homomorphism and in addition we sketch some ideas about how to identify those homomorphisms between subobject lattices which arise from arrows.

Adhesive categories [2] have been shown to provide a general categorical setting in which double-pushout rewriting can be defined in such a way that some fundamental results like the local Church-Rosser theorem and the concurrency theorem can be proved without the need for any additional conditions. The framework of adhesive categories encompasses graphs and several other graphical structures which play a role in the theory of concurrent and distributed systems.

It is a well-known and useful fact that the set of subobjects of an object in an adhesive category forms a distributive lattice. In lattice theory, distributive lattices have been extensively studied [1], but to our knowledge this theory has not been applied directly to adhesive categories. Here we review the relevant lattice-theoretic concepts and investigate in which way arrows of an adhesive category can be seen as lattice homomorphisms. We will restrict ourselves to the theory of finite distributive lattices, leaving the generalization to the infinite case as future work. Correspondingly, in several places, we will consider only finite objects, i.e., objects that have only finitely many subobjects.

Adhesive Categories. Adhesive categories have been introduced in [2] as categories where pushouts along monomorphisms are so-called Van-Kampen squares. For this paper the most important fact about adhesive categories is that the subobjects of an object form a distributive lattice. In addition to the usual requirements we here assume the existence of a strict initial object 0.

The Representation Theorem for Finite Distributive Lattices. We next review some lattice-theoretic concepts, following mainly the presentation in [1].

Hereafter, we will refer to a lattice $(L, \sqsubseteq)$ with meets and joins denoted by $\sqcap$ and $\sqcup$, and top and bottom elements, whenever they exist, denoted by $\bot$, $\top$.

In a lattice, an element a is called an *irreducible* whenever $a \neq \bot$ and $a = b \sqcup c$ implies $a = b$ or $a = c$. The set of all irreducibles of L is denoted by $\mathcal{J}(L)$.

For a finite lattice it can be easily seen that an element a is irreducible if and only if it has exactly one direct predecessor with respect to $\sqsubset$ (where $a \sqsubset b \iff (a \sqsubseteq b \wedge a \neq b)$). Furthermore in such a lattice every element b can be represented as a join of irreducibles as $b = \bigsqcup\{a \in \mathcal{J}(L) \mid a \sqsubseteq b\}$.

This leads us directly to Birkhoff's representation theorem for finite distributive lattices which says that every such lattice L is isomorphic to the lattice of downward-closed subsets of $\mathcal{J}(L)$, ordered by subset inclusion. The isomorphism maps every lattice element $a \in L$ to the set of irreducibles $\{x \in \mathcal{J}(L) \mid x \sqsubseteq a\}$.

Furthermore there is an important duality principle saying that every lattice homomorphism from L to K (which preserves in addition $\top$ and $\bot$) induces a monotone map from $\mathcal{J}(K)$ to $\mathcal{J}(L)$ and vice versa.

Subobject Lattices and Irreducibles. It is shown in [2] that subobjects of an object A in an adhesive category form a distributive lattice. Moreover, $0 \to A$ is the bottom and $id_A : A \to A$ is the top element.

The representation theorem for finite distributive lattices applies directly whenever A is finite. Furthermore, given an object I and two monos $I \to A$ and $I \to B$ we can show that $I \to A$ is an irreducible subobject of A if and only if $I \to B$ is an irreducible subobject of B. Hence a notion of irreducible object can be defined independently of any specific subobject lattice.

For the category of directed unlabelled graphs the irreducibles are the single node, the single edge and the loop, whereas for edge-labelled graphs irreducibles consist of nodes, single edges and loops for every label. If instead we consider graphs with second-order edges which connect two ordinary edges, also second-order edges must be among the irreducibles. This means that irreducibles may become arbitrarily "large" and arbitrarily "nested".

Arrows and Lattice Homomorphisms. We now discuss some facts about the relationship between arrows and lattice homomorphisms. Note that it is also one of the requirements of a Heyting category that arrows are lattice homomorphisms in the sense described below. This is a connection which has to be further studied.

Proposition 1 (Arrows are lattice homomorphisms). *In an adhesive category every arrow between objects induces a lattice homomorphism in the other direction via the retyping relation. That is, for an arrow $\varphi: A \to A'$ there is a lattice homorphism $\varphi^{-1}: Sub(A') \to Sub(A)$, where every subobject $b': B' \to A'$ of A' is mapped to a subobject $b: B \to A$ of A via the following pullback:*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b'} \\ A & \xrightarrow[\varphi]{} & A' \end{array}$$

Proof (Sketch). The retyping preserves joins due to the Van-Kampen square property of adhesive categories and meets due to standard pullback splitting results.

The fact that an arrow φ induces a lattice homomorphism η in the opposite direction resembles the duality principle stated above, with the arrow φ playing the role of the monotone map on irreducibles. However, not every lattice homomorphisms $\eta\colon Sub(A') \to Sub(A)$ can be seen as an arrow $A \to A'$.

For instance take two graphs both consisting only of a single edge, but with different labels (say X in the first and Y in the second graph). Then the two subobject lattices are isomorphic, but there is no morphism between these graphs.

Conjecture. Let A, A' be finite objects and let $\eta\colon Sub(A') \to Sub(A)$, be a lattice homomorphism. Let us fix some notation. Assume that any irreducible $i' \in Sub(A')$ is of the kind $i' : I' \to A'$ and its image, which might not be an irreducible, is $\eta(i') = b : B \to A$. Then there exists an arrow $\varphi : A \to A'$ such that $\varphi^{-1} = \eta$ if and only if

1. for any irreducible $i' \in Sub(A')$ there exists an arrow $\varphi_{i'}$: $B \to I'$;
2. given two irreducibles $i'_1, i'_2 \in Sub(A')$ such that $i'_1 \sqsubseteq i'_2$ (and thus $\eta(i'_1) \sqsubseteq \eta(i'_2)$), then the square on the right is a pullback, where the vertical arrows are the monos which witness the order relation between the corresponding subobjects.

$$\begin{array}{ccc} B_2 & \xrightarrow{\varphi_{i'_2}} & I'_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow[\varphi_{i'_1}]{} & I'_1 \\ \downarrow & & \downarrow \\ A & & A' \end{array}$$

Note that the conditions above are clearly necessary: Condition 1 follows immediately from the definition of φ^{-1} while Condition 2 follows from pullback splitting. The real challenge is to prove that they are also sufficient.

Conclusions. As future work we will focus on the conjecture above and on the generalization to infinite objects. We hope to be able to extend the characterization of lattice homomorphisms which are arrows to a representation theorem for objects in adhesive categories, in the spirit of Birkhoff's representation theorem. Finally it would be interesting to understand whether the assumption of having a strict initial object can be omitted without losing any of the results of interest.

References

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2. Stephen Lack and Paweł Sobociński. Adhesive and quasiadhesive categories. *RAIRO – Theoretical Informatics and Applications*, 39(3), 2005.