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Composition and Decomposition of DPO Transformations with Borrowed Context*

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Abstract. Double-pushout (DPO) transformations with borrowed context extend the standard DPO approach by allowing part of the graph needed in a transformation to be borrowed from the environment. The bisimilarity based on the observation of borrowed contexts is a congruence, thus facilitating system analysis. In this paper, focusing on the situation in which the states of a global system are built out of local components, we show that DPO transformations with borrowed context defined on a global system state can be decomposed into corresponding transformations on the local states and vice versa. Such composition and decomposition theorems, developed in the framework of adhesive categories, can be seen as a first step towards an inductive definition, in SOS style, of the labelled transition system associated to a graph transformation system. As a special case we show how an ordinary DPO transformation on a global system state can be decomposed into local DPO transformations with borrowed context using the same production.

1 Introduction

Graph transformations [6] have been applied successfully to several areas of software and system engineering, including syntax and semantics of visual languages, visual modelling of behaviour and programming, metamodelling and model transformation, refactoring of models and programs. Almost invariably the underlying idea is the same: the states of a system are modelled by suitable graphs and state changes are represented by graph transformations. Consequently, the behaviour of the system is expressed by a transition system, where states are reachable graphs and transitions are induced by graph transformations. The transition system can be the basis for defining various notions of abstract behavioural equivalences, e.g., trace, failures and bisimulation equivalence. These, in turn, can be used to provide a solid theoretical justification

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for various constructions and techniques in the above mentioned areas of system engineering, e.g., for the formalisation of behavioural refinement, or to show semantical correctness of refactoring and model transformation.

The applicability of these techniques generally requires the considered behavioural equivalence to be a congruence: two systems—seen as equivalent from the point of view of an external observer—must be equivalent also in all possible contexts or environments.

Unfortunately, behavioural equivalences defined over unlabelled transition systems naively generated by using transformation rules often fail to be congruences. The same problem arises for several other computational formalisms which can be naturally endowed with an operational semantics based on unlabelled reductions, such as the λ-calculus [1] or many process calculi with mobility or name passing, e.g., the π-calculus [11] or the ambient calculus [3].

In order to overcome this problem recently there has been a lot of interest in the automatic derivation of labelled transition systems where bisimilarity is a congruence for reactive systems endowed with an (unlabelled) reduction semantics (see, e.g., [10, 8, 5, 12]). In particular, in the case of double-pushout (DPO) graph rewriting this has led to an extension of the approach, called DPO approach with borrowed contexts [5]. Intuitively a label C of a transition represents the (minimal) context that must be "added" to the current state in order to allow the transformation or reduction step to be performed.

In this paper, we focus on the situation in which the states of a global system are built out of local states of the components of the systems. Then we show that DPO transformations with borrowed context defined on a global system state can be decomposed into corresponding transformations on the local states. Vice versa we study the conditions under which local transformations can be composed to yield global ones. The main results of this paper are composition and decomposition theorems for DPO transformations with borrowed context in the framework of rewriting systems over adhesive categories [9]. As a special case we show how an ordinary DPO transformation on a global system state can be decomposed into local DPO transformations with borrowed context using the same production.

These composition and decomposition results can be seen as a first step towards a structural operational semantics for adhesive rewriting systems, i.e., towards a framework where the transition system associated to a graph transformation system can be defined inductively, in SOS style. Compare for instance the inductive CCS rule stating that from $P \xrightarrow{a} P'$ and $Q \xrightarrow{\bar{a}} Q'$ (where a is an action and $\bar{a}$ its corresponding coaction) one can derive $P \mid Q \xrightarrow{\tau} P' \mid Q'$ (where the label τ stands for a silent transition). Intuitively $P \xrightarrow{a} P'$ means that P can move to Q' if the environment performs an output on channel a and, similarly, Q can move if the environment performs an input on a. The two local moves can be combined leading to a transition for $P \mid Q$ where nothing is "borrowed" from the environment (as expressed by the τ-label).

Having an inductive way of specifying the behaviour of a graph can lead to a new understanding of system semantics and new proof techniques. E.g., inductive definitions can be quite useful when comparing the semantics of two calculi, as in [2].

The rest of the paper is structured as follows. In Section 2 we introduce the basics of adhesive categories and of the DPO approach with borrowed contexts. In Section 3 we introduce a category of transformations with borrowed contexts, which is the basis for the formalisation of the composition and decomposition theorems for transformations given in Sections 4 and 5, respectively. Finally, in Section 6 we conclude and outline directions of future research.

2 DPO Transformation with Borrowed Contexts

Adhesive categories have been introduced in [9], as categories where pushouts along monomorphisms are so-called Van-Kampen squares (see Condition 3 in the definition below). We will only briefly sketch the theory of adhesive categories.

Definition 1 (Adhesive category). *A category* $\mathbf{C}$ *is called* adhesive *if*

1. $\mathbf{C}$ *has pushouts along monos;*
2. $\mathbf{C}$ *has pullbacks;*
3. *Given a cube diagram as shown on the right with: (i)* $A \to C$ *mono, (ii) the bottom square a pushout and (iii) the left and back squares pullbacks, we have that the top square is a pushout iff the front and right squares are pullbacks.*

$$
\begin{tikzcd}
A' \arrow[rr] \arrow[dr] \arrow[dd] & & C' \arrow[dr] \arrow[dd] & \\
& B' \arrow[rr] \arrow[dd] & & D' \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & C \arrow[dr] & \\
& B \arrow[rr] & & D
\end{tikzcd}
$$

The category $\mathbf{Set}$ of sets and functions is adhesive. Adhesive categories enjoy closure properties, for instance if $\mathbf{C}$ is adhesive then so is any functor category $\mathbf{C}^{\mathbf{X}}$, any slice category $\mathbf{C}{\downarrow}C$ and any co-slice category $C{\downarrow}\mathbf{C}$. Therefore, since the category of graphs and graph morphisms is a functor category $\mathbf{Graph} \cong \mathbf{Set}^{\bullet \leftleftarrows \bullet}$, it is adhesive.

A subobject of a given object T is an isomorphism class of monomorphisms to T. Binary intersections of subobjects exist in any category with pullbacks. In adhesive categories also binary unions of subobjects exist and can be obtained by taking the pushout over their intersection. Moreover, the lattice of subobjects is distributive.

Theorem 2 ([9]). *For an object* T *of an adhesive category* $\mathbf{C}$*, the partially ordered set* $Sub(T)$ *of subobjects of* T *is a distributive lattice. Given two subobjects* $A, B \in Sub(T)$*, the meet* $A \cap B$ *is (the isomorphism class of) their pullback, while the join* $A \cup B$ *is (the isomorphism class of) their pushout in* $\mathbf{C}$ *over their* *intersection.*

$$\begin{tikzcd}[row sep=small]
& A \arrow[dr] & \\
A \cap B \arrow[ur] \arrow[dr] \arrow[rr, phantom, "PB"] & & T \\
& B \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
& A \arrow[dr] \arrow[drr, bend left] & & \\
A \cap B \arrow[ur] \arrow[dr] \arrow[rr, phantom, "PO"] & & A \cup B \arrow[r, dotted] & T \\
& B \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

The following lemma will be useful in the future where we have to show that certain squares in adhesive categories are pullbacks or pushouts. It follows directly from Theorem 2.

Lemma 3. *Consider the following diagram where all arrows are mono. The square below is a pullback if and only if $A = B \cap C$ (all objects are seen as subobjects of E). Furthermore the square is a pushout if and only if $A = B \cap C$ and $D = B \cup C$.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] & \\
C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

We next define rewriting with borrowed contexts on objects (e.g., over graphs) with interfaces, as introduced in [5]. Intuitively, the borrowed context is the smallest extra context which must be added to the object being rewritten in order to obtain an occurrence of the left-hand side. The extra context can be added only using the interface.

Definition 4 (Borrowed contexts, transformations). *Let* $\mathbf{C}$ *be a fixed adhesive category and let $r = (L \leftarrow I \rightarrow R)$ be a rewriting rule. A* DPO transformation with borrowed context*—short* transformation*—t (of r) is a diagram in* $\mathbf{C}$ *of the following form, where all arrows are mono:*

$$\begin{tikzcd}
D \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & L \arrow[d] \arrow[dr, phantom, "PO"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & R \arrow[d] \\
G \arrow[r] \arrow[dr, phantom, "PO"] & G^+ \arrow[dr, phantom, "PB"] & C \arrow[l] \arrow[r] & H \\
J \arrow[r] \arrow[u] & F \arrow[u] & K \arrow[l] \arrow[u] \arrow[ur, dotted] &
\end{tikzcd}$$

In this case we write $(J \rightarrow G) \overset{r,m}{\Longrightarrow} (K \rightarrow H)$ where $m = G \leftarrow D \rightarrow L$ is the partial match*. If instead we want to focus on the interaction with the environment we say that $J \rightarrow G$ makes a* transition *with borrowed context $J \rightarrow F \leftarrow K$ and becomes $K \rightarrow H$ (written: $(J \rightarrow G) \overset{J \rightarrow F \leftarrow K}{\longrightarrow} (K \rightarrow H)$).*

For a given transformation t_i we will denote the objects occurring in the corresponding diagram by D_i, G_i, J_i, G_i^+, C_i, H_i, F_i, K_i.

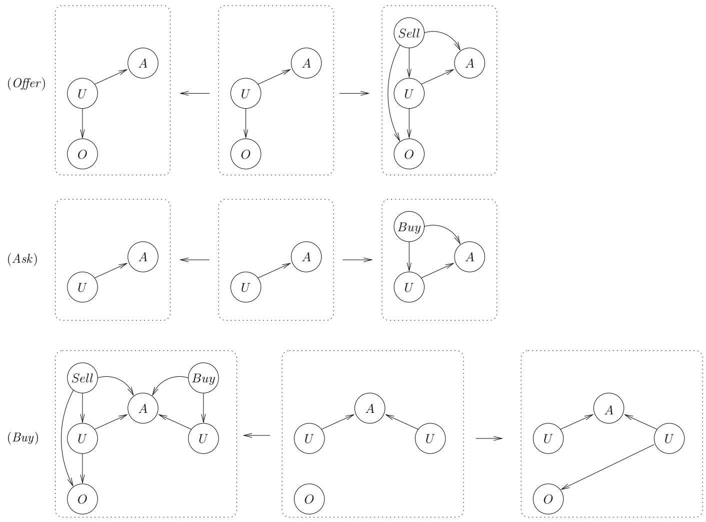

Fig. 1. Rewriting system Market.

The squares in the diagram above have the following meaning: the upper left-hand square merges the left-hand side L and the object G to be rewritten according to a partial match $G \leftarrow D \rightarrow L$ of the left-hand side in G. The resulting object G^+ contains a total match of L and can be rewritten as in the standard DPO approach, which produces the two remaining squares in the upper row. The pushout in the lower row gives us the borrowed (or minimal) context F which is missing in order to obtain a total match of L, along with a morphism $J \rightarrow F$ indicating how F should be attached to G. Finally, the interface for the resulting object H is obtained by "intersecting" the borrowed context F and the object C via a pullback. Roughly, the new interface includes what is preserved of the old interface and of the context borrowed from the environment. The two pushout complements that are constructed in Definition 4 may not exist. In this case no rewriting step is possible.

It has been shown in [5] that bisimilarity on the transition system labelled with borrowed contexts is a congruence with respect to cospan composition.

Example. Consider the category **Graph** of labelled graphs and label-preserving morphisms. Take the rewriting system Market in **Graph** depicted in Fig. 1, which can be interpreted as a very high-level description of the interactions between users of an electronic market place. Graph nodes are represented as circles, with their label inside. Edges are directed and unlabelled. Users, represented as U-labelled nodes, can possess objects, denoted by O-labelled nodes, and they can be connected to one (or more) market places, represented by A-labelled nodes.

A user possessing some objects can autonomously decide to offer one of them to other users, on a market place, expressed by rule *(Offer)*. A user can also ask

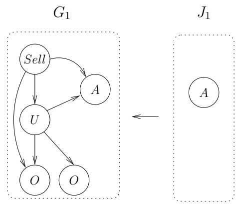

Fig. 2. The graph with interface $J_1 \to G_1$.

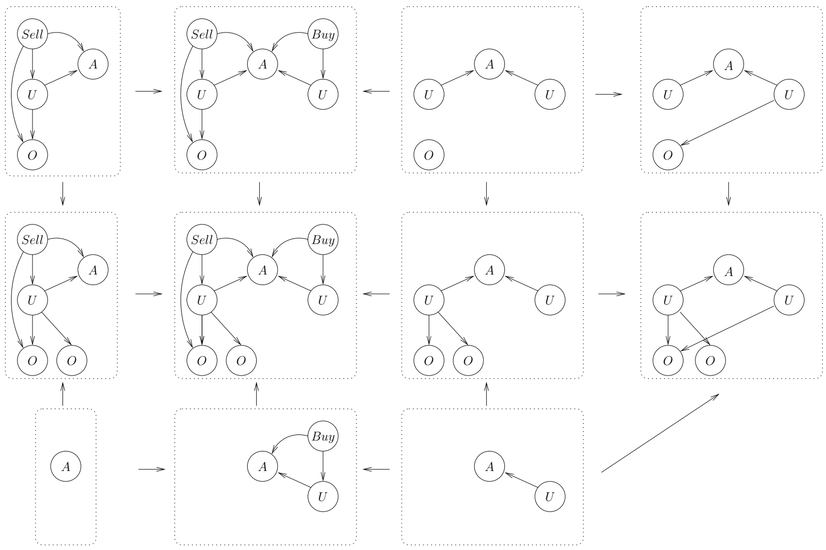

Fig. 3. A transformation with borrowed context t_1 over $J_1 \to G_1$, using rule *(Buy)*.

for something to buy on a market he is connected to, expressed by rule *(Ask)*. A request and an offer, after some negotiation which is not modelled, can meet, the object is sold and moved from the seller to the buyer, modelled by rule *(Buy)*.

An example of a transformation with borrowed context using production *(Buy)* can be found in Fig. 3. It is applied to the graph with interface $J_1 \to G_1$ in Fig. 2. The graph G_1 includes a market place A, with a user U, possessing two objects and trying to sell one of them. Note that the borrowed context consists of an additional user playing the role of a buyer. In other words, the existence of the transformation expresses the fact that rule *(Buy)*, can be applied assuming that the context provides a user which buys the object sold by the user in G_1.

Remark: Note that we obtain the well-known case of DPO transformations if we consider total matches $L \to G$ instead of partial matches $G \leftarrow D \to L$, which implies $G = G^+$. In this case we can take any interface object J , for instance the initial object—if it exists in the category—which implies that F and K are also initial objects.

3 Transformation Morphisms

A first step towards the composition of transformations is the formalisation of the intuitive idea of embedding of a transformation into another. This is done by introducing a suitable notion of transformation morphism.

Definition 5 (Transformation morphisms). *Let t_1, t_2 be two transformations for a fixed rewriting rule $L \leftarrow I \rightarrow R$. A* transformation morphism *$\theta\colon t_1 \to t_2$ consists of arrows $D_1 \to D_2$, $G_1 \to G_2$, $G_1^+ \to G_2^+$, $C_1 \to C_2$, $H_1 \to H_2$, $J_2 \to J_1$, $F_2 \to F_1$ and $K_2 \to K_1$ such that the diagram below commutes. (The arrows $L \to L$, $I \to I$, $R \to R$ in the diagram are the identities.)*

A transformation morphism is called componentwise mono *if it is composed of monos only.*

$$
\begin{tikzcd}
D_2 \arrow[rrr] \arrow[dr] & & & L \arrow[dr] & & & I \arrow[lll] \arrow[rrr] \arrow[dr] & & & R \arrow[dr] & \\
& G_2 \arrow[rrr] & & & G_2{}^+ & & & C_2 \arrow[lll] \arrow[rrr] & & & H_2 \\
& & J_2 \arrow[ul] \arrow[rrr] \arrow[ddd] & & & F_2 \arrow[ul] \arrow[ddd] & & & K_2 \arrow[lll] \arrow[ul] \arrow[ddd] & & \\
D_1 \arrow[uuu] \arrow[rrr] \arrow[dr] & & & L \arrow[uuu] \arrow[dr] & & & I \arrow[uuu] \arrow[lll] \arrow[rrr] \arrow[dr] & & & R \arrow[uuu] \arrow[dr] & \\
& G_1 \arrow[uuu] \arrow[rrr] & & & G_1{}^+ \arrow[uuu] & & & C_1 \arrow[uuu] \arrow[lll] \arrow[rrr] & & & H_1 \arrow[uuu] \\
& & J_1 \arrow[ul] \arrow[rrr] & & & F_1 \arrow[ul] & & & K_1 \arrow[lll] \arrow[ul] & &
\end{tikzcd}
$$

The intuition—at least if all arrows are mono—is that a morphism "embeds" transformation t_1 into t_2. Thus, G_1 (the object being rewritten) is mapped into G_2 and the same holds for D_1 (the partial match), G_1^+, C_1 and H_1. Furthermore, since G_1 is contained in G_2, it might be necessary to borrow more context from the environment. Hence F_1 can be larger than F_2 and the same holds for the inner and outer interfaces of F_1 (denoted by J_1 and K_1). For instance J_1 might have to be larger than J_2 since more context has to be attached. Hence the "squares" J_2, J_1, G_1, G_2 and F_2, F_1, G_1^+, G_2^+ and K_2, K_1, C_1, C_2 are not real squares, but will be called *horseshoes* in the following.

The complexity of our proofs stems from the fact that these horseshoes have to be taken into account. Otherwise it would be possible to simply work in a functor category.

Definition 6 (Category of transformations). *The category having as objects transformations and as arrows transformation morphisms is denoted by* **Trafo**. *Composition of transformation morphisms is defined componentwise.*

Example. Consider the graph with interface $J_3 \to G_3$ in Fig. 4. The graph G_3 includes a market place with two users. The first one possesses two objects and

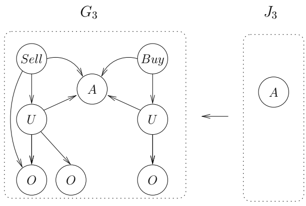

Fig. 4. The graph with interface $J_3 \to G_3$.

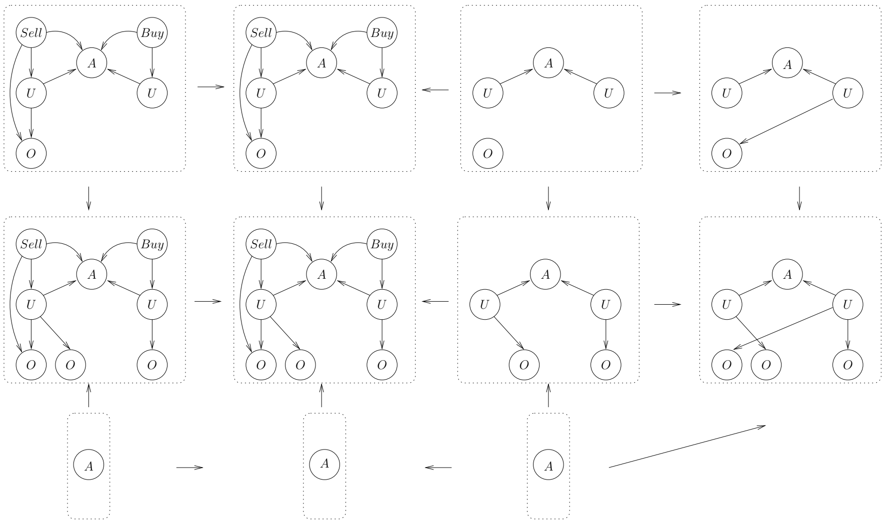

Fig. 5. A transformation with borrowed context t_3 over $J_3 \to G_3$, using rule *(Buy)*.

is trying to sell one of them. The second user is looking for an object to buy. A transformation for $J_3 \to G_3$, using rule *(Buy)*, can be found in Fig. 5. Observe that in this case the given graph already includes all what is needed for applying rule *(Buy)* and thus nothing is actually borrowed from the context. Thus only the interface is exposed in the label, i.e., the graph $F_3 = J_3$. It is not difficult to see that there is an obvious transformation morphism $\theta_1 : t_1 \to t_3$, where t_1 is the transformation in Fig. 3.

Although the definition of transformation morphisms does not impose any condition on the vertical squares or horseshoes, we can infer some properties by taking into account that all horizontal squares are either pullbacks or pushouts (along monos, and thus also pullbacks).

Lemma 7 (Properties of transformation morphisms). *For a transformation morphism θ as defined in Definition 5 it holds that:*

- *The squares* I, I, L, L *and* I, I, R, R *and* C_1, C_2, G_1^+, G_2^+ *and* C_1, C_2, H_1, H_2 *are pushouts.*
- *If the arrows* $G_1^+ \to G_2^+$, $C_1 \to C_2$ *and* $H_1 \to H_2$ *are mono, the squares* L, L, G_1^+, G_2^+ *and* I, I, C_1, C_2 *and* R, R, H_1, H_2 *and* K_2, K_1, F_2, F_1 *and* D_1, D_2, G_1, G_2 *are pullbacks.*

4 Composition of Transformations

In this section we study a composition mechanism for transformations. More precisely we show that given two transformations t_1, t_2, using the same production, with a common subtransformation t_0, the two transformations can be combined via a pushout. We will give sufficient conditions for the existence of this pushout and show how it can be constructed.

We first consider a simpler category where objects are pushouts and we show how to construct pushouts in this setting.

Lemma 8 (Pushouts in the category of pushouts). *Let* $\mathbf{C}$ *be a fixed adhesive category. Consider the category of pushouts in* $\mathbf{C}$*, where objects are pushouts* p_i *of the form*

$$\begin{tikzcd} A_0^i \arrow[r] \arrow[d] & A_2^i \arrow[d] \\ A_1^i \arrow[r] & A_3^i \end{tikzcd}$$

and an arrow $\varphi: p_1 \to p_2$ *consists of four arrows* $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ *(with* $\varphi_i: A_i^1 \to A_i^2$*) which connect the corners of the squares such that the full diagram (which is a cube) commutes.*

Given two arrows $\varphi^1: p_0 \to p_1$, $\varphi^2: p_0 \to p_2$ *in this category, a pushout* $\psi^1: p_1 \to p_3$, $\psi^2: p_2 \to p_3$ *can be computed by constructing four pushouts of the arrows* φ_i, ψ_i*, provided that these pushouts exist. Then the resulting (pushout) square is composed of the four mediating arrows.*

Note that even if the pushouts p_0, p_1, p_2 consist only of monos, the resulting pushout square p_3 does not necessarily consist of monos. Hence in our case this property has to be shown by different means.

We next introduce a property ensuring the composability of transformations.

Definition 9 (Composable transformation morphisms). *Let* $\theta_i: t_0 \to t_i$ *with* $i \in \{1, 2\}$ *be transformation morphisms. We say that* θ_1 *and* θ_2 *are* composable *if*

1. θ_1, θ_2 *are componentwise mono* and
2. *the square in the underlying category* $\mathbf{C}$ *in Fig. 6(a) (where the top and right arrows appear in* θ_1 *and the left and bottom arrows appear in* θ_2*) is a pullback.*

$$
\begin{tikzcd}
D_0 \arrow[r] \arrow[d] & D_1 \arrow[d] \\
D_2 \arrow[r] & L
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
& t_1 \arrow[dr, "\theta'_1"] & \\
t_0 \arrow[ur, "\theta_1"] \arrow[dr, "\theta_2"'] & & t_3 \\
& t_2 \arrow[ur, "\theta'_2"'] &
\end{tikzcd}
$$

(b)

Fig. 6. Composition of transformations.

Intuitively the second condition in the definition above requires that the partial match for t_0 is the intersection of the partial matches for t_1, t_2.

Theorem 10 (Composition of transformations). *Let $\theta_i : t_0 \to t_i$ with $i \in \{1, 2\}$ be two composable transformation morphisms. Then the pushout of θ_1, θ_2 exists (see Fig. 6(b)) and can be obtained in the following way:*

- *Construct $D_3, G_3, G_3^+, C_3, H_3$ by taking pushouts and J_3, F_3, K_3 by taking pullbacks. For instance D_3 is constructed by taking the pushout of $D_0 \to D_1$, $D_0 \to D_2$, where these two arrows are taken from θ_1 respectively θ_2. This produces the transformation morphisms $\theta'_i : t_i \to t_3$.*
- *In order to construct the arrows in t_3 we proceed as follows:*
  - *Most arrows can be immediately obtained as mediating arrows. This is the case for $D_3 \to G_3, D_3 \to L, G_3 \to G_3^+, L \to G_3^+, C_3 \to G_3^+, C_3 \to H_3, J_3 \to F_3, K_3 \to F_3$, $I \to C_3$, $R \to H_3$.*
  - *Furthermore construct $J_3 \to G_3$ by composing $J_3 \to J_1 \to G_1 \to G_3$. Similarly for $F_3 \to G_3^+$ and $K_3 \to C_3$.*

5 Decomposition of Transformations

In the previous section we have shown how to compose larger transformations out of smaller ones. Here we are going into the opposite direction and show under which conditions transformations can be split into smaller ones. That is, given a transformation of $J \to G$ and a decomposition of G into subobjects G_1, G_2, is it possible to find transformations for these subobjects, such that the composition of these transformations yields the original transformation?

5.1 Projecting Transformations

In order to be able to formulate the decomposition of transformations, we will first show how to project a transformation to a subobject of G, i.e., to a subobject of the object to be rewritten. We identify some conditions which ensure that a transformation can be projected over a subobject of the rewritten object. Roughly, the interface of the subobject must be sufficiently large to guarantee that the needed context can be actually borrowed.

Definition 11 (Extensibility). *Let t_2 be a transformation and and let $J_2 \to J_1 \to G_1 \to G_2$ be a factorisation of the arrow $J_2 \to G_2$. Then the transformation is called* extensible *with respect to this factorisation, whenever there exists a subobject F_1 of U_2 (the pushout of $G_2^+ \leftarrow C_2 \to H_2$) such that*

$$G_1 \cup L = G_1 \cup F_1 \qquad G_1 \cap F_1 = J_1.$$

The definition above basically requires (in lattice-theoretic terms) that the pushout complement F_1 of $J_1 \to G_1 \to G_1^+$ exists, where $G_1^+ = G_1 \cup L$. Note that in adhesive categories the pushout complement of monos is unique (if it exists).

The extensibility condition given in Definition 11 can be difficult to work with. Below we give an alternative handier condition, sufficient for extensibility.

Lemma 12 (Sufficient condition for extensibility). *Let t_3 be a transformation and let $J_3 \to J_1 \to G_1 \to G_3$ be a factorisation of the arrow $J_3 \to G_3$. Then t_3 is extensible with respect to this factorisation if the pushout complement X_{13} of $J_1 \to G_1 \to G_3$ exists, i.e., there exists an object X_{13} and morphisms such that the square below is a pushout.*

$$\begin{tikzcd} J_1 \arrow[r] \arrow[d] & G_1 \arrow[d] \\ X_{13} \arrow[r] & G_3 \end{tikzcd}$$

In this case set $F_1 = (X_{13} \cup F_3) \cap (G_1 \cup L)$.

Essentially, the sufficient condition requires that the interface of the smaller object G_1 is sufficiently large to allow to get the larger object G_3 by extending G_1 along its interface.

Now let t_i be a transformation over an object with interface $J_i \to G_i$ ($i \in \{1, 2\}$) and let $J_2 \to J_1 \to G_1 \to G_2$ be a factorisation of $J_2 \to G_2$. We say that a transformation morphism $\theta : t_1 \to t_2$ is *consistent* with the factorisation if it has the arrows $J_2 \to J_1$ and $G_1 \to G_2$ as components.

Proposition 13 (Projection of transformations). *Let t_2 be a transformation and let $J_2 \to J_1 \to G_1 \to G_2$ be a (mono) factorisation of the morphism $J_2 \to G_2$ such that t_2 is extensible with respect to this factorisation. Then there exists a unique transformation t_1 of $J_1 \to G_1$, with a componentwise mono transformation morphism $\theta\colon t_1 \to t_2$, consistent with the factorisation.*

The objects of this transformation can be constructed as follows:

1. *Construct U_2 as the pushout of $C_2 \to G_2^+$ and $C_2 \to H_2$. Now all objects can be considered as subobjects of U_2.*
2. *The object F_1 is given by the extensibility property above, which requires that $G_1 \cup L = G_1 \cup F_1$ and $G_1 \cap F_1 = J_1$. Set $D_1 = G_1 \cap D_2$, $G_1^+ = G_1 \cup L$, $C_1 = G_1^+ \cap C_2$, $H_1 = C_1 \cup R$, $K_1 = F_1 \cap C_1$.*

5.2 Decomposing Transformations

As a first step towards the decomposition of a transformation, we introduce a suitable decomposition for an object with interface.

Definition 14 (Proper decomposition). *Let $J_3 \to G_3$ be an object with interface. Then a* proper decomposition *of $J_3 \to G_3$ is a cube as shown below where all arrows are mono, the square G_0, G_1, G_2, G_3 is a pushout and and the square J_0, J_1, J_2, J_3 is a pullback. (Note that the four remaining "squares" are horseshoes.)*

$$\begin{tikzcd}
J_0 \arrow[dd, tail] & & J_2 \arrow[ll, tail] \arrow[dd, tail] & \\
& J_1 \arrow[ul, tail] \arrow[dd, tail] & & J_3 \arrow[ul, tail] \arrow[ll, tail] \arrow[dd, tail] \\
G_0 \arrow[rr, tail] \arrow[dr, tail] & & G_2 \arrow[dr, tail] & \\
& G_1 \arrow[rr, tail] & & G_3
\end{tikzcd}$$

Theorem 15 (Decomposition of transformations). *Let t_3 be a transformation of an object with interface $J_3 \to G_3$. Consider a proper decomposition of $J_3 \to G_3$ as in Definition 14 and assume that the transformation t_3 is extensible with respect to the factorisations $J_3 \to J_1 \to G_1 \to G_3$ and $J_3 \to J_2 \to G_2 \to G_3$.*

Then there are transformations t_i for $J_i \to G_i$ (where $i \in \{0, 1, 2\}$) with componentwise mono transformation morphisms $\theta_j : t_0 \to t_j$, $\theta'_j : t_j \to t_3$ (where $j \in \{1, 2\}$) forming a pushout in the category of transformations (see the diagram in Theorem 10). These transformation morphisms can be obtained via projections as described in Proposition 13.

Observe that, if in the cube in Theorem 15 above we have the special (but very typical) case where $J_0 = J_1 = J_2 = J_3 = G_0$ (and all arrows between these objects are the identities), the sufficient extensibility condition of Lemma 12 is satisfied: in the terminology of this lemma $X_{13} = G_2$ and $X_{23} = G_1$.

In a sense, composition and decomposition are inverse to each other up to isomorphism. The fact that composition is the inverse of decomposition has been shown directly in Theorem 15. On the other hand, since projections are unique (by Proposition 13), there is—up to isomorphism—only one way to decompose a transformation according to a proper decomposition of the rewritten object (see Definition 14). Hence, also decomposition is the inverse of composition.

Next we discuss the special case where a DPO rewriting step with trivial borrowed context is decomposed, leading to transformations with possibly non-empty borrowed contexts. Assume that $G = G_3$ can be split into G_0, G_1, G_2 as in the pushout diagram below on the left and consider a DPO rewriting step for G_3. Then this step can be extended to a transformation with borrowed context for G_3 (with interface G_0) with a total match of the left-hand side.

$$
\begin{tikzcd}
 & G_0 \arrow[dl] \arrow[dr] & \\
G_1 \arrow[dr] \arrow[rr, phantom, "\mathrm{PO}"] & & G_2 \arrow[dl] \\
 & G_3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & L \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & R \arrow[d] \\
G \arrow[r] & G & C \arrow[l] \arrow[r] & H \\
G_0 \arrow[r] \arrow[u] \arrow[ur, phantom, "\mathrm{PO}"] & G_0 \arrow[u] \arrow[ur, phantom, "\mathrm{PB}"] & G'_0 \arrow[l] \arrow[u] \arrow[ur, dotted] &
\end{tikzcd}
$$

In this case we can set $J_0 = J_1 = J_2 = J_3 = G_0$ and obtain a proper decomposition of $J_3 \to G_3$ as in Definition 14 (the top square is trivially pullback and the bottom square is a pushout by assumption). Then, decomposing transformation t_3 as described in Proposition 15 leads to three transformations t_0, t_1, t_2, with—in general—partial matches D_0, D_1, D_2.

Example. Consider the graph with interface $J_3 \to G_3$ in Fig. 4. Note that there is an obvious factorisation $J_3 \to J_1 \to G_1 \to G_3$ of $J_3 \to G_3$. Furthermore, the transformation t_3 in Fig. 5, which uses rule *(Buy)*, is extensible along such factorisation. In fact, the sufficient condition given by Lemma 12 is satisfied.

Therefore we can project the transformation t_3 in Fig. 5 along such factorisation thus obtaining the transformation t_1 over $J_1 \to G_1$ depicted in Fig. 3. As already noted, as an effect of projecting the transformation over a smaller graph, the borrowed context becomes non-trivial (larger than the interface): the rule can be applied assuming that the context provides a user which buys the object sold by the user in G_1.

More generally, consider the diagram in Fig. 8, where morphisms are the inclusions suggested by the shapes of the graphs. This is a pushout in **Graph**. Moreover, we can imagine all graphs G_i to have an interface given by $J_i = G_0$. Then the conditions of Proposition 15 are satisfied: we can project the transformation t_3 in Fig. 5 to transformations over $J_i \to G_i$ ($i \in \{0, 1, 2\}$). The projection over $J_1 \to G_1$ leads to the transformation t_1 in Fig. 3, while the projection over $J_2 \to G_2$ leads to the transformation t_2 in Fig. 7. Both t_1 and t_2 project to the same derivation t_0 over $J_0 \to G_0$. The pushout of the obtained transformations can be computed, according to Theorem 10 to obtain t_3 again.

6 Conclusion and Comparison to Related Work

In this paper, focusing on a setting in which a system is built out of smaller components, we discussed how derivations with borrowed context over the global state can be decomposed into transformations over the local state of each single component using the same rule. Vice versa, we showed that, under suitable consistency conditions, local transformations can be composed to give rise to a transformation over the global system state.

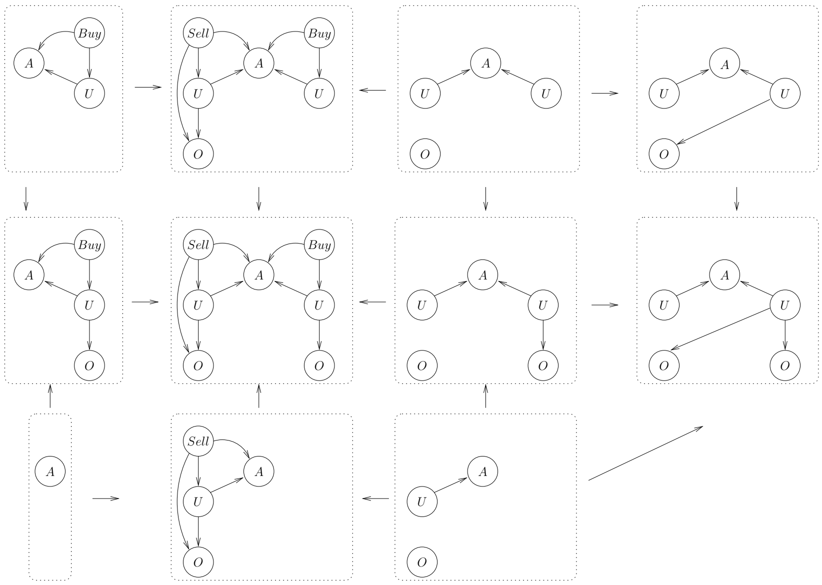

Fig. 7. A transformation with borrowed context t_2 over $J_2 \to G_2$, using rule *(Buy)*.

We remark that the form of composition described in this paper is quite different from amalgamation as described for instance in [4]. There two transformations for different rules are amalgamated producing a transformation for the amalgamated rule. In our case, instead, the rule is fixed and the transformations differ with respect to the context that has to be borrowed from the environment. By composing objects and hence transformations we obtain additional structure which might reduce the borrowed context.

The composition and decomposition results can be seen as a basic step towards the possibility of defining transformations only for "atomic objects" and assemble all possible transformations out of these atomic transformations, and thus, towards an inductive definition, in SOS style, of the transition system of a graph transformation system (more generally an adhesive rewriting system). In addition to the composition result we will also need the possibility to compose an evolving object with a passive context and to have rules for handling restrictions of the interface. This would correspond to the communication, parallel composition and restriction rules for process calculi. Additionally, composition would be even more natural and closer to process calculi if performed over so-called rewriting steps, hiding the internal details, rather than on full transformations. That is—in the terminology of Definition 4—we would like to observe only the object with interface $J \to G$, the resulting object $K \to H$ and the label or borrowed context $J \to F \leftarrow K$, but not the objects D, G^+, C which are only auxiliary or intermediate objects. We plan to extend our approach to this setting.

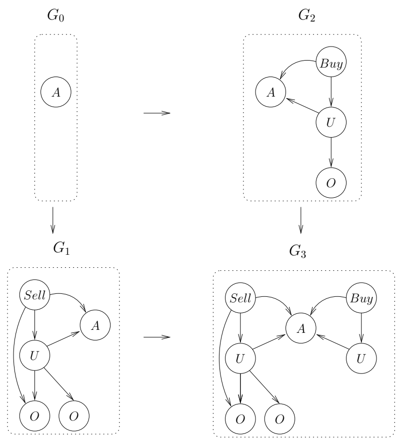

Fig. 8. Decomposition of transformations.

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A Proofs

We will now give the proofs of all lemmas, propositions and theorems in the paper. First we will start with some useful lemmas.

Lemma 16. *In any distributive lattice it holds that:*

- *Whenever $A \cap C \subseteq B$ and $B \cap C \subseteq A$, then $(A \cup B) \cap C = A \cap B \cap C$.*
- *Whenever $B \subseteq A \cup C$ and $A \subseteq B \cup C$, then $(A \cap B) \cup C = A \cup B \cup C$.*

Lemma 17. *Consider a transformation t as in Definition 4. Take the pushout of the arrows $C \to G^+$, $C \to H$ and denote the pushout object by U. Now consider all objects as subobjects of U.*

Then it holds that $F \subseteq L \cup J$ and $C \subseteq G \cup I$.

Proof. We will only show $F \subseteq L \cup J$, the other inclusion can be shown analogously. We know that $F \subseteq G^+$. Hence we have by Lemma 3:

$$F = F \cap G^+ = F \cap (G \cup L) = (F \cap G) \cup (F \cap L) = J \cup (F \cap L) \subseteq J \cup L.$$

Lemma 7. *For a transformation morphism θ as defined in Definition 5 it holds that:*

- *The squares I, I, L, L and I, I, R, R and C_1, C_2, G_1^+, G_2^+ and C_1, C_2, H_1, H_2 are pushouts.*
- *If the arrows $G_1^+ \to G_2^+$, $C_1 \to C_2$ and $H_1 \to H_2$ are mono, the squares L, L, G_1^+, G_2^+ and I, I, C_1, C_2 and R, R, H_1, H_2 and K_2, K_1, F_2, F_1 and D_1, D_2, G_1, G_2 are pullbacks.*

Proof.

Pushout I, I, L, L: obvious

Pushout I, I, R, R: obvious

Pushout C_1, C_2, G_1^+, G_2^+: Follows from the fact that the top, bottom and back squares of the middle cube in the back row are pushouts.

Pushout C_1, C_2, H_1, H_2: Follows from the fact that the top, bottom and back squares of the right cube in the back row are pushouts.

Pullback L, L, G_1^+, G_2^+: Follows from the fact that $G_1^+ \to G_2^+$ is a mono.

Pullback I, I, C_1, C_2: Since $C_1 \to C_2$ is mono.

Pullback R, R, H_1, H_2: Since $H_1 \to H_2$ is mono.

Pullback K_2, K_1, F_2, F_1: Follows from the fact that the top, bottom and back squares of the right cube in the front row are pullbacks.

Pullback D_1, D_2, G_1, G_2: Follows from the fact that the top, right and bottom squares of the left cube in the back row are pullbacks.

Lemma 8. (Pushouts in the category of pushouts) *Let* $\mathbf{C}$ *be a fixed adhesive category. Consider the category of pushouts in* $\mathbf{C}$*, where objects are pushouts* p_i *of the form*

$$\begin{array}{ccc} A_0^i & \longrightarrow & A_2^i \\ \downarrow & & \downarrow \\ A_1^i & \longrightarrow & A_3^i \end{array}$$

and an arrow $\varphi\colon p_1 \to p_2$ *consists of four arrows* $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ *(with* $\varphi_i\colon A_i^1 \to A_i^2$*) which connect the corners of the squares such that the full diagram (which is a cube) commutes.*

Given two arrows $\varphi^1\colon p_0 \to p_1$*,* $\varphi^2\colon p_0 \to p_2$ *in this category, a pushout* $\psi^1\colon p_1 \to p_3$*,* $\psi^2\colon p_2 \to p_3$ *can be computed by constructing four pushouts of the arrows* φ_i*,* ψ_i*, provided that these pushouts exist. Then the resulting (pushout) square is composed of the four mediating arrows.*

Proof.

The proof is a direct consequence of the 4-cube pushout lemma [7] which says that in the four-dimensional cube in Figure 9 parallel pushouts $A_0^0, A_1^0, A_2^0, A_3^0$ and $A_0^1, A_1^1, A_2^1, A_3^1$ and $A_0^2, A_1^2, A_2^2, A_3^2$ and $A_0^0, A_0^1, A_0^2, A_0^3$ and $A_1^0, A_1^1, A_1^2, A_1^3$ and $A_2^0, A_2^1, A_2^2, A_2^3$, then $A_0^3, A_1^3, A_2^3, A_3^3$ is a pushout if and only if $A_3^0, A_3^1, A_3^2, A_3^3$ is a pushout.

Since the squares $A_0^i, A_1^i, A_2^i, A_3^i$ for $i \in \{0,1,2,3\}$ are given as pushouts and the objects A_j^3 are constructed as pushout objects in $A_j^0, A_j^1, A_j^2, A_j^3$ for $j \in \{0,1,2,3\}$ we can conclude that $A_0^3, A_1^3, A_2^3, A_3^3$ is a pushout as well. The 4-cube commutes by construction of pushouts in a diagram category.

Theorem 10. (Composition of transformations) *Let* $\theta_i\colon t_0 \to t_i$ *with* $i \in \{1,2\}$ *be two composable transformation morphisms. Then the pushout of* θ_1*,* θ_2 *exists (see Fig. 6(b)) and can be obtained in the following way:*

- *Construct* $D_3, G_3, G_3^+, C_3, H_3$ *by taking pushouts and* J_3, F_3, K_3 *by taking pullbacks. For instance* D_3 *is constructed by taking the pushout of* $D_0 \to D_1$*,* $D_0 \to D_2$*, where these two arrows are taken from* θ_1 *respectively* θ_2*. This produces the transformation morphisms* $\theta_i'\colon t_i \to t_3$*.*
- *In order to construct the arrows in* t_3 *we proceed as follows:*
  - *Most arrows can be immediately obtained as mediating arrows. This is the case for* $D_3 \to G_3, D_3 \to L, G_3 \to G_3^+, L \to G_3^+, C_3 \to G_3^+, C_3 \to H_3, J_3 \to F_3, K_3 \to F_3$*,* $I \to C_3$*,* $R \to H_3$*.*
  - *Furthermore construct* $J_3 \to G_3$ *by composing* $J_3 \to J_1 \to G_1 \to G_3$*. Similarly for* $F_3 \to G_3^+$ *and* $K_3 \to C_3$*.*

Proof.
Commutativity:

The four-dimensional cubes resulting from the three upper squares in a transformation with borrowed contexts (see diagram in Definition 4) commute by Lemma 8. The 4-cubes for the two lower squares contain several “horseshoe”

$$
\begin{tikzcd}
A_0^1 \arrow[rrrr] \arrow[drrr] \arrow[dddddd] & & & & A_2^1 \arrow[drrr] \arrow[dddddd] & & & \\
& & & A_0^3 \arrow[rrrr] \arrow[dddddd] & & & & A_2^3 \arrow[dddddd] \\
& A_0^0 \arrow[uul] \arrow[rrrr] \arrow[dr] \arrow[dd] & & & & A_2^0 \arrow[uul] \arrow[dr] \arrow[dd] & & \\
& & A_0^2 \arrow[uur] \arrow[rrrr] \arrow[dd] & & & & A_2^2 \arrow[uur] \arrow[dd] & \\
& A_1^0 \arrow[rrrr] \arrow[dr] \arrow[ddl] & & & & A_3^0 \arrow[dr] \arrow[ddl] & & \\
& & A_1^2 \arrow[rrrr] \arrow[ddr] & & & & A_3^2 \arrow[ddr] & \\
A_1^1 \arrow[rrrr] \arrow[drrr] & & & & A_3^1 \arrow[drrr] & & & \\
& & & A_1^3 \arrow[rrrr] & & & & A_3^3
\end{tikzcd}
$$

Fig. 9. A four-dimensional cube.

squares, their commutativity can be shown by straightforward diagram chasing techniques.

Upper squares are pushouts:
From Lemma 8 it follows that by the construction above we obtain three pushouts: D_3, G_3, L, G_3^+ and I, L, C_3, G_3^+ and I, R, C_3, H_3 (i.e., the three squares shown below).

$$\begin{tikzcd}
D_3 \arrow[r] \arrow[d] & L \arrow[d] & I \arrow[l] \arrow[r] \arrow[d] & R \arrow[d] \\
G_3 \arrow[r] & G_3^+ & C_3 \arrow[l] \arrow[r] & H_3
\end{tikzcd}$$

All arrows in t_3 are monos:
We will now show that all arrows in transformation t_3 are monos:

$D_3 \to G_3$**:** In order to show this consider the cube below. Since mediating arrows from mono pushouts to mono pullbacks are mono in adhesive categories, it suffices to check that the square D_0, D_1, D_2, G_3 is a pullback.

$$\begin{tikzcd}
D_0 \arrow[rr] \arrow[dr] \arrow[dd] & & D_2 \arrow[dr] \arrow[dd] & \\
& D_1 \arrow[rr] \arrow[dd] & & D_3 \arrow[dd] \\
G_0 \arrow[rr] \arrow[dr] & & G_2 \arrow[dr] & \\
& G_1 \arrow[rr] & & G_3
\end{tikzcd}$$

The upper and lower squares are pushouts by construction of D_3 and G_3 and the back and left squares are pullbacks by 7. From this and the VK square property it follows that the front and right squares are pullbacks.
Now consider the following diagram with an object D' and arrows $D' \to D_1$, $D' \to D_2$ that make the outer diagram commute. Note that the lower two triangles are pullbacks and the upper triangle is a pushout (and hence also a pullback). In order to obtain a mediating arrow $D' \to D_0$ we have to show that $D' \xrightarrow{\psi} D_1 \to D_3 = D' \to D_2 \to D_3$.
We have $D' \xrightarrow{\psi} D_1 \to G_1 \to G_3 = D' \to D_2 \to G_2 \to G_3 = D' \to D_2 \to D_3 \to G_3$. Since the left lower triangle is a pullback, it follows that there exists a unique arrow $\varphi\colon D' \to D_1$ such that $D' \xrightarrow{\varphi} D_1 \to G_1 = D' \xrightarrow{\psi} D_1 \to G_1$ and $D' \xrightarrow{\varphi} D_1 \to D_3 = D' \to D_2 \to D_3$. Since $D_1 \to G_1$ is a mono, we

can conclude that $\varphi = \psi$ and hence $D' \xrightarrow{\psi} D_1 \to D_3 = D' \to D_2 \to D_3$.

$$\begin{tikzcd}
 & & D' \arrow[d, dotted] \arrow[ddll, "\psi"'] \arrow[ddrr] & & \\
 & & D_0 \arrow[dll] \arrow[drr] & & \\
D_1 \arrow[rr] \arrow[dr] & & D_3 \arrow[dd] & & D_2 \arrow[ll] \arrow[dl] \\
 & G_1 \arrow[dr] & & G_2 \arrow[dl] & \\
 & & G_3 & &
\end{tikzcd}$$

$D_3 \to L$**:** This arrow is a mediating arrow in the following diagram, where the inner square is a pushout and the outer square is a pullback by assumption. Hence the mediating arrow is a mono.

$$\begin{tikzcd}
 & D_1 \arrow[dr] \arrow[drr] & & \\
D_0 \arrow[ur] \arrow[dr] & & D_3 \arrow[r, dotted, tail] & L \\
 & D_2 \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

$L \to G_3^+$**,** $I \to C_3$**,** $R \to H_3$**:** These three arrows are mediating arrows of the following diagrams, where the outer squares are clearly pullbacks. Hence all these arrows are mono.

$$\begin{tikzcd}
 & L \arrow[dr] \arrow[drr] & & \\
L \arrow[ur] \arrow[dr] & & L \arrow[r, dotted, tail] & G_3^+ \\
 & L \arrow[ur] \arrow[urr] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I \arrow[dr] \arrow[drr] & & \\
I \arrow[ur] \arrow[dr] & & I \arrow[r, dotted, tail] & C_3 \\
 & I \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & R \arrow[dr] \arrow[drr] & & \\
R \arrow[ur] \arrow[dr] & & R \arrow[r, dotted] & H_3 \\
 & R \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

$G_3 \to G_3^+$, $C_3 \to G_3^+$, $C_3 \to H_3$: These arrows are monos since monos are preserved by pushouts in adhesive categories.

$J_3 \to F_3$, $K_3 \to F_3$: We have that $J_3 \to F_3 \to F_1 = J_3 \to J_1 \to F_2$ (this follows from the fact that $J_3 \to F_3$ is a mediating arrow of the diagram below). Since $J_3 \to J_1 \to F_1$ is mono, we can conclude that $J_3 \to F_3$ is also mono. Similarly we can show that $K_3 \to F_3$ is mono.

$$
\begin{tikzcd}
 & J_1 \arrow[r] & F_1 \arrow[dr] & \\
J_3 \arrow[ur] \arrow[r, dotted] \arrow[dr] & F_3 \arrow[ur] \arrow[dr] & & F_0 \\
 & J_2 \arrow[r] & F_2 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K_1 \arrow[r] & F_1 \arrow[dr] & \\
K_3 \arrow[ur] \arrow[r, dotted] \arrow[dr] & F_3 \arrow[ur] \arrow[dr] & & F_0 \\
 & K_2 \arrow[r] & F_2 \arrow[ur] &
\end{tikzcd}
$$

$J_3 \to G_3$, $F_3 \to G_3^+$, $K_3 \to C_3$: We have that $J_3 \to G_3 = J_3 \to J_1 \to G_1 \to G_3$, i.e., it can be factorised into monos and is hence a mono. Analogously we can show that $F_3 \to G_3^+$ and $K_3 \to C_3$ are monos.

Squares are pullbacks or pushouts:

We now have to show that t_3 is a transformation, i.e., all squares in t_3 commute and the squares are either pushouts or pullbacks as demanded by the definition of a transformation. In order to show the latter we construct U_3 as the pushout of $C_3 \to G_3^+$ and $C_3 \to H_3$ and consider every object as a subobject of U_3. This can be done since all arrows are mono.

Furthermore we will exploit Lemmas 3 and 16.

We now examine the remaining squares. It is already clear from the remarks above that the upper squares D_3, G_3, L, G_3^+ and I, L, C_3, G_3^+ and I, R, C_3, H_3 are pushouts.

Pushout J_3, G_3, F_3, G_3^+: We start by observing that $J_3 = J_1 \cap J_2 = (G_1 \cap F_1) \cap (G_2 \cap F_2) = G_1 \cap G_2 \cap (F_1 \cap F_2) = G_1 \cap G_2 \cap F_3$ and we have to show that this is equal to $G_3 \cap F_3 = (G_1 \cup G_2) \cap F_3$. According to Lemma 16 this holds whenever $G_1 \cap F_3 \subseteq G_2$ and $G_2 \cap F_3 \subseteq G_1$. We only show the first inclusion, the second one can be shown analogously: $G_1 \cap F_3 = G_1 \cap F_1 \cap F_2 = J_1 \cap F_2 \subseteq J_1 \subseteq J_0 \subseteq G_0 \subseteq G_2$. This implies $J_3 = G_3 \cap F_3$.

Now let us show $G_3 \cup F_3 = G_3^+$: We have $G_3^+ = G_1^+ \cup G_2^+ = (G_1 \cup F_1) \cup (G_2 \cup F_2) = (G_1 \cup G_2) \cup (F_1 \cup F_2) = G_3 \cup F_1 \cup F_2$. We have to show that this is equal to $G_3 \cup F_3 = G_3 \cup (F_1 \cap F_2)$. According to Lemma 16 it is sufficient to show $F_1 \subseteq G_3 \cup F_2$ and $F_2 \subseteq G_3 \cup F_1$. Again we only show the first inclusion: $F_1 \subseteq G_1^+ = G_1 \cup L \subseteq G_1 \cup G_2^+ = G_1 \cup G_2 \cup F_2 = G_3 \cup F_2$.

Pullback K_3, C_3, F_3, G_3^+: We start by observing that $K_3 = K_1 \cap K_2 = (C_1 \cap F_1) \cap (C_2 \cap F_2) = C_1 \cap C_2 \cap (F_1 \cap F_2) = C_1 \cap C_2 \cap F_3$ and we have to show that this is equal to $C_3 \cap F_3 = (C_1 \cup C_2) \cap F_3$. According to Lemma 16 this holds whenever $C_1 \cap F_3 \subseteq C_2$ and $C_2 \cap F_3 \subseteq C_1$. We only show the first inclusion, the second one can be shown analogously: $C_1 \cap F_3 = C_1 \cap F_1 \cap F_2 = K_1 \cap F_2 \subseteq K_1 \subseteq K_0 \subseteq C_0 \subseteq C_2$.

Pushout property:

Finally we have to make sure that t_3 with morphisms $\theta_i' \colon t_i \to t_3$ is the pushout of θ_1, θ_2. So let t' be another transformation with morphisms $\xi_i \colon t_i \to t'$.

We can construct a mediating morphism $t_3 \to t'$ by taking the mediating arrows of all the pushouts and pullbacks.

$$
\begin{tikzcd}
& t_1 \arrow[dr, "\theta_1'"] \arrow[drr, bend left, "\xi_1"] & & \\
t_0 \arrow[ur, "\theta_1"] \arrow[dr, "\theta_2"'] & & t_3 \arrow[r, dotted, "\xi"] & t' \\
& t_2 \arrow[ur, "\theta_2'"] \arrow[urr, bend right, "\xi_2"'] & &
\end{tikzcd}
$$

In order to show commutativity we only pick three typical cases. All other cases can be handled analogously.

Commutativity of the square D_3, D', G_3, G': Commutativity of this square follows directly from Lemma 8.

Commutativity of the horseshoe J', J_3, G_3, G':

$$
\begin{tikzcd}
& J' \arrow[rrr, bend left] \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & & & G' & \\
& J_3 \arrow[rrr, bend left] \arrow[dl] \arrow[dr] & & & G_3 \arrow[u] & \\
J_1 \arrow[rrr, bend left] \arrow[dr] & & J_2 \arrow[rrr, bend left] \arrow[dl] & G_1 \arrow[uur, bend left] \arrow[ur] & & G_2 \arrow[ul] \arrow[uul, bend right] \\
& J_0 \arrow[rrr, bend left] & & & G_0 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

We have (because of the commutativity of the horseshoe J_3, J_1, G_1, G_3):

$$
J' \to J_3 \to G_3 \to G' = J' \to J_3 \to J_1 \to G_1 \to G_3 \to G' = J' \to J_1 \to G_1 \to G' \\
= J' \to G'.
$$

Commutativity of the square J', J_3, F', F_3:

$$\begin{tikzcd}
& J' \arrow[rrr, bend left] \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & & & F' \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& J_3 \arrow[rrr, bend left] \arrow[dl] \arrow[dr] & & & F_3 \arrow[dl] \arrow[dr] & \\
J_1 \arrow[rrr, bend left] \arrow[dr] & & J_2 \arrow[rrr, bend left] \arrow[dl] & F_1 \arrow[dr] & & F_2 \arrow[dl] \\
& J_0 \arrow[rrr, bend left] & & & F_0 &
\end{tikzcd}$$

It follows from the commutativity of the squares J_3, J_1, F_3, F_1 and J', J_1, F', F_1 that

$$\begin{aligned} &J' \to J_3 \to F_3 \to F_1 = J' \to J_3 \to J_1 \to F_1 = J' \to J_1 \to F_1 \\ = &J' \to F' \to F_1 = J' \to F' \to F_3 \to F_1. \end{aligned}$$

Then we can conclude that $J' \to J_3 \to F_3 = J' \to F' \to F_3$ since $F_3 \to F_1$ is mono.

Proposition 13. (Projection of transformations) *Let t_2 be a transformation and let $J_2 \to J_1 \to G_1 \to G_2$ be a (mono) factorisation of the morphism $J_2 \to G_2$ such that t_2 is extensible with respect to this factorisation. Then there exists a unique transformation t_1 of $J_1 \to G_1$, with a componentwise mono transformation morphism $\theta: t_1 \to t_2$, consistent with the factorisation.*

The objects of this transformation can be constructed as follows:

1. *Construct U_2 as the pushout of $C_2 \to G_2^+$ and $C_2 \to H_2$. Now all objects can be considered as subobjects of U_2.*
2. *The object F_1 is given by the extensibility property above, which requires that $G_1 \cup L = G_1 \cup F_1$ and $G_1 \cap F_1 = J_1$. Set $D_1 = G_1 \cap D_2$, $G_1^+ = G_1 \cup L$, $C_1 = G_1^+ \cap C_2$, $H_1 = C_1 \cup R$, $K_1 = F_1 \cap C_1$.*

Proof.

Uniqueness:

Uniqueness follows from the fact that D_1, D_2, G_1, G_2 must be a pullback and C_1, C_2, G_1^+, G_2^+ must be a pushout (see Lemma 7) and that a transformation t_1 is uniquely determined by D_1, G_1 and J_1 in the case where $\theta: t_1 \to t_2$ is componentwise mono.

Existence of vertical arrows:

We have to check that all arrows necessary for the transformation morphism do in fact exist. For this is it is sufficient to show inclusion of the subobjects (this also automatically gives us commutativity):

$D_1 \to D_2$**:** $D_1 = G_1 \cap D_2 \subseteq D_2$
$G_1^+ \to G_2^+$**:** $G_1^+ = G_1 \cup L \subseteq G_2 \cup L = G_2^+$
$C_1 \to C_2$**:** $C_1 = G_1^+ \cap C_2 \subseteq C_2$
$H_1 \to H_2$**:** $H_1 = C_1 \cup R \subseteq C_2 \cup R = H_2$
$F_2 \to F_1$**:** In order to show that $F_2 \subseteq F_1$ we will check that $F_2 = F_1 \cap F_2$. Since F_2 is the pushout complement of $J_2 \to G_2 \to G_2^+$ it is that unique object that satisfies $G_2 \cap F_2 = J_2$ and $G_2 \cup F_2 = G_2^+$. It remains to show that these conditions are also true for $F_1 \cap F_2$, i.e., $G_2 \cap F_1 \cap F_2 = J_2$ and $G_2 \cup (F_1 \cap F_2) = G_2^+$.

Using $J_2 = F_2 \cap G_2$ we have that $F_1 \cap F_2 \cap G_2 = F_1 \cap J_2$ and since $J_2 \subseteq J_1 \subseteq F_1$ we have that $F_1 \cap F_2 \cap G_2 = F_1 \cap J_2 = J_2$.

Furthermore $(F_1 \cap F_2) \cup G_2 = (F_1 \cup G_2) \cap (F_2 \cup G_2) = (F_1 \cup G_2) \cap G_2^+$. So in order to show that this is equal to G_2^+ we have to show that $G_2^+ \subseteq F_1 \cup G_2$. Since $G_2^+ = L \cup G_2$ it is sufficient to show that $L \subseteq F_1 \cup G_2$, which follows from $L \subseteq L \cup G_1 = F_1 \cup G_1 \subseteq F_1 \cup G_2$, using the extensibility property for F_1.

$K_2 \to K_1$**:** Since $K_1 = F_1 \cap C_1$ we have to show that $K_2 \subseteq F_1$ and $K_2 \subseteq C_1$. The first inclusion holds since $K_2 \subseteq F_2 \subseteq F_1$. Furthermore we have $K_2 \subseteq F_1 \subseteq G_1 \cup L = G_1^+$ by the extensibility property and $K_2 \subseteq C_2$ and hence $K_2 \subseteq G_1^+ \cap C_2 = C_1$.

Horizontal squares are pullbacks or pushouts:

We have to check that the five squares of transformation t_1 are pullbacks or pushouts as required in Definition 4 (at the same time we will show the existence of the necessary horizontal arrows between objects).

Pushout D_1, G_1, L, G_1^+**:** We first show that $G_1 \cap L = D_1$. First, observe that $D_1 = G_1 \cap D_2 \subseteq G_1 \cap L$. On the other hand $G_1 \subseteq G_2$ and $D_2 = G_2 \cap L$ and hence $G_1 \cap L \subseteq G_1 \cap G_2 \cap L = G_1 \cap D_2 = D_1$.

Furthermore $G_1^+ = G_1 \cup L$ by definition.

Pushout I, C_1, L, G_1^+**:** We have that $C_1 \cap L = G_1^+ \cap C_2 \cap L = G_1^+ \cap I = I$, since $I \subseteq L \subseteq G_1^+$.

Furthermore $C_1 \cup L = (G_1^+ \cap C_2) \cup L = (G_1^+ \cup L) \cap (C_2 \cup L) = G_1^+ \cap G_2^+ = G_1^+$ (since $L \subseteq G_1^+ \subseteq G_2^+$).

Pushout I, C_1, R, H_1**:** We have that $C_1 \cap R = (G_1^+ \cap C_2) \cap R = G_1^+ \cap I = I$, since $I \subseteq L \subseteq G_1^+$.

Finally $C_1 \cup R = H_2$ holds by definition.

Pushout J_1, G_1, F_1, G_1^+**:** This square is a pushout by the extensibility requirement.

Pullback K_1, C_1, F_1, G_1^+**:** $K_1 = F_1 \cap C_1$ holds by definition.

Theorem 15. (Decomposition of transformations) *Let t_3 be a transformation of an object with interface $J_3 \to G_3$. Consider a proper decomposition of $J_3 \to G_3$ as in Definition 14 and assume that the transformation t_3 is extensible with respect to the factorisations $J_3 \to J_1 \to G_1 \to G_3$ and $J_3 \to J_2 \to G_2 \to G_3$.*

Then there are transformations t_i for $J_i \to G_i$ (where $i \in \{0,1,2\}$) with componentwise mono transformation morphisms $\theta_j\colon t_0 \to t_j$, $\theta'_j\colon t_j \to t_3$ (where $j \in \{1,2\}$) forming a pushout in the category of transformations (see the diagram in Theorem 10). These transformation morphisms can be obtained via projections as described in Proposition 13.

Proof.

Existence of pushout complements:

The objects F_1, F_2 exist due to the extensibility requirement. However, we first have to check that there exists an object F_0 which is the pushout complement $J_0 \to G_0 \to G_0^+ = G_0 \cup L$.

We set $F_0 = F_1 \cup F_2 \cup J_0$ and check that it satisfies the necessary conditions:

Pushout complement F_0: We want to show $G_0 \cap F_0 = J_0$ and $G_0 \cup F_0 = G_0 \cup L$. We have that $F_0 \cap G_0 = (F_1 \cup F_2 \cup J_0) \cap G_0 = (F_1 \cap G_0) \cup (F_2 \cap G_0) \cup (J_0 \cap G_0) = (F_1 \cap G_1 \cap G_2) \cup (F_2 \cap G_1 \cap G_2) \cup J_0 = (J_1 \cap G_2) \cup (J_2 \cap G_1) \cup J_0 \subseteq J_1 \cup J_2 \cup J_0$. But equality obviously follows since $J_0 \subseteq (J_1 \cap G_2) \cup (J_2 \cap G_1) \cup J_0$. Furthermore $F_0 \cup G_0 = F_1 \cup F_2 \cup J_0 \cup G_0 = F_1 \cup F_2 \cup (G_1 \cap G_2) = (F_1 \cup F_2 \cup G_1) \cap (F_1 \cup F_2 \cup G_2) = (F_2 \cup G_1^+) \cap (F_1 \cup G_2^+)$. It holds that $F_2 \subseteq F_1 \subseteq G_1^+$ and similarly $F_1 \subseteq G_2^+$. Hence $(F_2 \cup G_1^+) \cap (F_1 \cup G_2^+) = G_1^+ \cap G_2^+ = G_0^+$.

Projection to the interface is unique:

Projecting t_3 first to $J_1 \to G_1$ and then to $J_0 \to G_0$ gives the same transformation as projecting via $J_2 \to G_2$. This follows from the uniqueness of the projection shown in Proposition 13.

Vertical squares are pullbacks or pushouts:

In order to show that the diagrams is in fact a pushout of transformation morphisms, we have to check that the vertical squares are pullbacks or pushouts as described in Proposition 10. As in Proposition 13 we can assume that all objects are subobjects of U_3 (the pushout of $C_3 \to G_3^+$ and $C_3 \to H_3$).

Pushout D_0, D_1, D_2, D_3: We have that $D_1 \cap D_2 = (G_1 \cap D_3) \cap (G_2 \cap D_3) = (G_1 \cap G_2) \cap D_3 = G_0 \cap D_3 = D_0$.
Furthermore $D_1 \cup D_2 = (G_1 \cap D_3) \cup (G_2 \cap D_3) = (G_1 \cup G_2) \cap D_3 = G_3 \cap D_3 = D_3$.

Pushout $G_0^+, G_1^+, G_2^+, G_3^+$: It holds that $G_1^+ \cap G_2^+ = (G_1 \cup L) \cap (G_2 \cup L) = (G_1 \cap G_2) \cup L = G_0 \cup L = G_0^+$.
Furthermore $G_1^+ \cup G_2^+ = (G_1 \cup L) \cup (G_2 \cup L) = (G_1 \cup G_2) \cup L = G_3 \cup L = G_3^+$.

Pushout C_0, C_1, C_2, C_3: We have that $C_1 \cap C_2 = (G_1^+ \cap C_3) \cap (G_2^+ \cap C_3) = (G_1^+ \cap G_2^+) \cap C_3 = G_0^+ \cap C_3 = C_0$.
Additionally $C_1 \cup C_2 = (G_1^+ \cap C_3) \cup (G_2^+ \cap C_3) = (G_1^+ \cup G_2^+) \cap C_3 = G_3^+ \cap C_3 = C_3$.

Pushout H_0, H_1, H_2, H_3: It holds that $H_1 \cap H_2 = (C_1 \cup R) \cap (C_2 \cup R) = (C_1 \cap C_2) \cup R = C_0 \cup R = H_0$.
Furthermore $H_1 \cup H_2 = (C_1 \cup R) \cup (C_2 \cup R) = (C_1 \cup C_2) \cup R = C_3 \cup R = H_3$.

Pullback F_3, F_2, F_1, F_0**:** We have to show that $F_1 \cap F_2 = F_3$. The borrowed context F_3 is included in F_1 and F_2 (this follows from the existence of transformation morphisms) and hence it is sufficient to show $F_1 \cap F_2 \subseteq F_3$.
Since $F_1 \subseteq G_1^+ \subseteq G_3^+$ and $F_2 \subseteq G_2^+ \subseteq G_3^+$ it holds that $F_1 \cap F_2 = (F_1 \cap F_2) \cap G_3^+ = (F_1 \cap F_2) \cap (G_3 \cup F_3) = (F_1 \cap F_2 \cap G_3) \cup (F_1 \cap F_2 \cap F_3)$. It is clear that $F_1 \cap F_2 \cap F_3$ is included in F_3. We have to show the same for $F_1 \cap F_2 \cap G_3$. Here it holds that $F_1 \cap F_2 \cap G_3 = F_1 \cap F_2 \cap (G_1 \cup G_2) = (F_1 \cap F_2 \cap G_1) \cup (F_1 \cap F_2 \cap G_2) = (J_1 \cap F_2) \cup (J_2 \cap F_1)$. Since $J_1 \subseteq J_0 \subseteq G_0 \subseteq G_2$ we have that $J_1 \cap F_2 \subseteq G_2 \cap F_2 = J_2$ and similarly $J_2 \cap F_1 \subseteq J_1$. Therefore $J_1 \cap F_2$ as well as $J_2 \cap F_1$ is contained in $J_1 \cap J_2$. Hence $(J_1 \cap F_2) \cup (J_2 \cap F_1) \subseteq J_1 \cap J_2 = J_3 \subseteq F_3$.

Pullback K_3, K_2, K_1, K_0**:** We have that $K_1 \cap K_2 = F_1 \cap C_1 \cap F_2 \cap C_2 = F_3 \cap C_0$. Now in order to show that $F_3 \cap C_0 = K_3$ we show both inclusions: first, $K_3 \subseteq F_3$ and $K_3 \subseteq K_0 \subseteq C_0$ which implies $K_3 \subseteq F_3 \cap C_0$. And finally $F_3 \cap C_0 \subseteq F_3 \cap C_3 = K_3$.

Lemma 12. (Sufficient Condition for Extensibility) *Let t_3 be a transformation and let $J_3 \to J_1 \to G_1 \to G_3$ be a factorisation of the arrow $J_3 \to G_3$. Then t_3 is extensible with respect to this factorisation if the pushout complement X_{13} of $J_1 \to G_1 \to G_3$ exists, i.e., there exists an object X_{13} and morphisms such that the square below is a pushout.*

$$\begin{tikzcd} J_1 \arrow[r] \arrow[d] & G_1 \arrow[d] \\ X_{13} \arrow[r] & G_3 \end{tikzcd}$$

In this case set $F_1 = (X_{13} \cup F_3) \cap (G_1 \cup L)$.

Proof. We have to check that F_1 satisfies $F_1 \cup G_1 = G_1 \cup L$ and $F_1 \cap G_1 = J_1$. We start by proving the first equation: $F_1 \cup G_1 = ((X_{13} \cup F_3) \cap (G_1 \cup L)) \cup G_1 = (X_{13} \cup G_1 \cup F_3) \cap (G_1 \cup L) = (G_3 \cup F_3) \cap (G_1 \cup L) = G_3^+ \cap (G_1 \cup L) = G_1 \cup L$.

Furthermore $F_1 \cap G_1 = ((X_{13} \cup F_3) \cap (G_1 \cup L)) \cap G_1 = (X_{13} \cup F_3) \cap G_1 = (X_{13} \cap G_1) \cup (F_3 \cap G_1) = J_1 \cup (F_3 \cap G_1)$. So it is left to show that $F_3 \cap G_1$ is contained in J_1: $F_3 \cap G_1 \subseteq F_3 \cap G_3 = J_3 \subseteq J_1$.